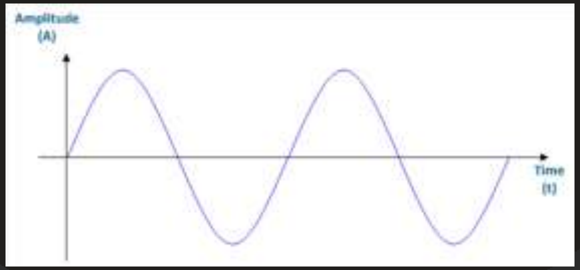
MODULE 1 INTRODUCTION TO DIGITAL SIGNAL PROCESSING



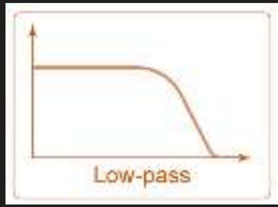
Analog

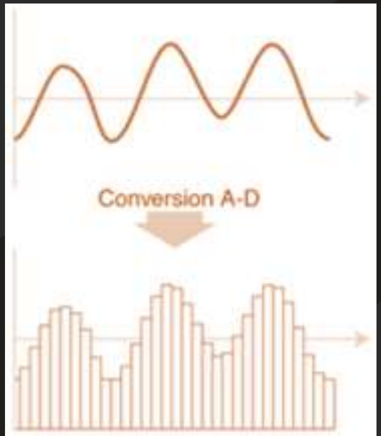
Digital

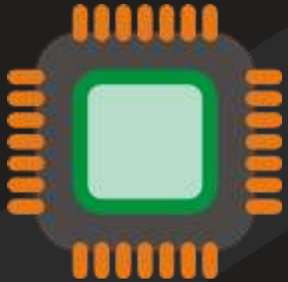
- Analysis, synthesis and modify
- Analog signal processing
- Digital signal processing

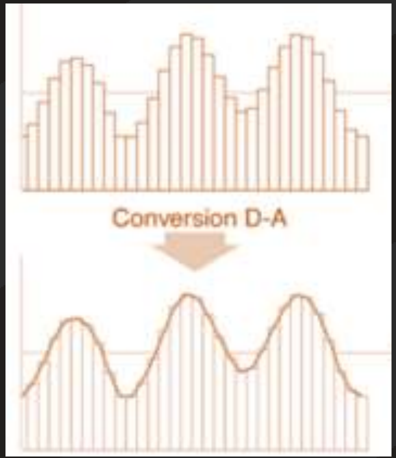
BASIC BLOCKS OF DIGITAL SIGNAL PROCESSING

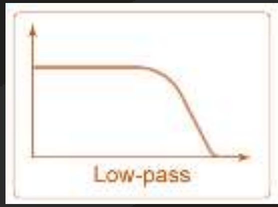
Analog input → Pre filter → AD Converter → Digital Signal Processor → DA Converter → Post filter → Analog output











Discrete Fourier Transform (DFT)

- DFT is a powerful computation tool which allows us to evaluate the Fourier transform on a digital computer or specifically designed hardware
- We notate like this

$$X(k) = DFT[x(n)] \qquad x(n) = IDFT[X(k)]$$

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi kn}{N}} \text{ , } 0 \le k \le N-1$$

IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{\frac{j2\pi kn}{N}} \text{ , } 0 \le n \le N-1$$

Let us define a term $W_N = e^{-\frac{j2\pi}{N}}$ which is known as twiddle factor and substitute in above equations

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk} \text{ , } 0 \le k \le N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)w_N^{-nk} \text{ , } 0 \le n \le N-1$$

Let us take an example

Q) Find the DFT of the sequence $x(n) = \{1,1,0,0\}$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi kn}{N}} \ ,0 \le k \le N-1$$

$$N = 4$$

$$X(k) = \sum_{n=0}^{3} x(n)e^{-\frac{j2\pi kn}{4}} \ ,0 \le k \le N-1$$

$k = 0$

$$X(0) = \sum_{n=0}^{3} x(n)e^{-\frac{j2\pi 0n}{4}}$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 1 + 0 + 0 \quad = 2$$

$k = 1$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-\frac{j\pi n}{2}}$$

$$= x(0) + x(1)e^{-\frac{j\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-\frac{j3\pi}{2}}$$

$$= 1 + 1\left[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right] + 0 + 0 \quad = 1 - j$$

$k = 2$

$$X(2) = \sum_{n=0}^{3} x(n)e^{-\frac{j\pi 2n}{2}}$$

$$= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= 1 + 1[\cos\pi - j\sin\pi] + 0 + 0 \quad = 1 - 1 = 0$$

$k = 3$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-\frac{j\pi 3n}{2}}$$

$$= x(0) + x(1)e^{-\frac{j3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{-\frac{j9\pi}{2}}$$

$$= 1 + 1\left[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right] + 0 + 0 \quad = 1 + j$$

$$X(k) = \{2, 1-j, 0, 1+j\}$$

DFT as linear transformation (Matrix method)

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{W_N^{nk}} \ , 0 \le k \le N-1$$

$$W_N = e^{-\frac{j2\pi}{N}}$$

Lets put *n = 0,1,2, … N-1*

$$X(k) = x(0).1 + x(1).W_N^{1k} + x(2).W_N^{2k} + \cdots + x(N-1)W_N^{(N-1)k}$$

$k = 0$

$$X(0) = x(0) + x(1) + x(2) + \cdots + x(N-1)$$

$k = 1$

$$X(1) = x(0) + x(1).W_N^1 + x(2).W_N^2 + \cdots + x(N-1)W_N^{(N-1)}$$

$\vdots$

$k = N-1$

$$X(N-1) = x(0) + x(1).W_N^{(N-1)} + x(2).W_N^{2(N-1)} + \cdots + x(N-1)W_N^{(N-1)(N-1)}$$

We can also represent the equation in matrix format

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & W_N^3 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & W_N^6 & \cdots & W_N^{2(N-1)} \\ 1 & W_N^3 & W_N^6 & W_N^9 & \cdots & W_N^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & W_N^{3(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ \vdots \\ x(N-1) \end{bmatrix}$$

$$X_N = W_N x_N$$

$$x_N = W_N^{-1} X_N$$

$$W_N^{-1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & \ddots & W_N^{-(N-1)} \\ \vdots & \vdots & & \\ 1 & W_N^{-(N-1)} & \cdots & W_N^{-(N-1)(N-1)} \end{bmatrix}$$

IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)w_N^{-nk} \ , 0 \le n \le N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left(w_N^{nk}\right)^*$$

Symbolically we can write as

$$x(n) = \frac{1}{N} X_N W_N^*$$

Comparing we get

$$W_N^{-1} = \frac{1}{N} W_N^*$$

Twiddle factor matrix

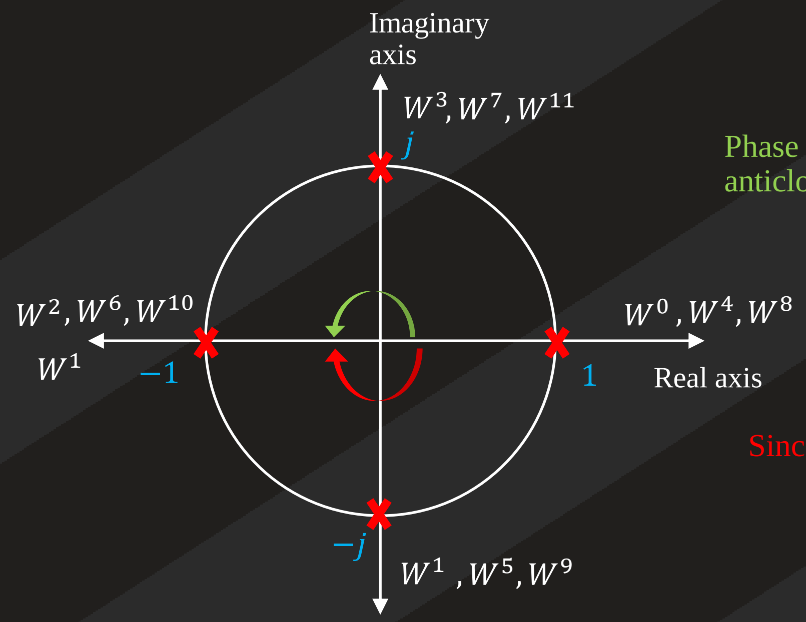
$$W_N = e^{-\frac{j2\pi}{N}}$$

Lies on the unit circle in the complex plane from 0 to 2π angle and it gets repeated for every cycle

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$W_2 = \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$



Phase change (0^0 - 360^0) - anticlockwise

Since $e^{-j\theta}$ – clockwise

Relationship of the DFT to Fourier Transform

Fourier-Transform

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x(n)e^{-j\omega n}$$

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi kn}{N}}$$

Comparing the above equations we get to find that DFT of x(n) is a sampled version of the FT of the sequence

$$X(k) = X(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{N}}, \qquad k = 0,1,2,\ldots N-1$$

Relationship between DFT & Fourie Transform

Relationship of the DFT to Z-Transform

Z-Transform

$$X(Z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}}$$

Substitute the value of x(n)

$$X(Z) = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}}\right] z^{-n}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left[\sum_{n=0}^{N-1} e^{\frac{j2\pi kn}{N}} z^{-n}\right]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left[\sum_{n=0}^{N-1}\left(e^{\frac{j2\pi k}{N}} z^{-1}\right)^{n}\right]$$

$$\sum_{k=0}^{N-1} a^{n} = \frac{1-\mathrm{a}^{\mathrm{N}}}{1-\mathrm{a}}$$

$$X(z) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left[\frac{1-\left(e^{\frac{j2\pi k}{N}} z^{-1}\right)^{N}}{1-e^{\frac{j2\pi k}{N}} z^{-1}}\right]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left[\frac{1-e^{j2\pi k} z^{-N}}{1-e^{\frac{j2\pi k}{N}} z^{-1}}\right]$$

In the above condition $e^{j2\pi k} = 1$ for all the values of k

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left[\frac{1-z^{-N}}{1-e^{\frac{j2\pi k}{N}} z^{-1}}\right]$$

$$X(z) = \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\left[\frac{X(k)}{1-e^{\frac{j2\pi k}{N}} z^{-1}}\right]$$

Relationship between DFT & Z- Transform

Properties of Discrete Fourier Transform

Periodicity

If X(k) is N-point DFT of a finite duration sequences x(n) then

$$x(n+N) = x(n) \quad for\ all\ n$$
$$X(k+N) = X(k) \quad for\ all\ k$$

Linearity

If two finite sequences $x_1(n)$ and $x_2(n)$ are linearly combined as

$$x_3(n) = ax_1(n) + bx_2(n)$$

Then DFT of the sequence

$$X_3(k) = aX_1(k) + bX_2(k)$$

$$ax_1(n) + bx_2(n) \overset{DFT}{\Longleftrightarrow} aX_1(k) + bX_2(k)$$

Circular time shift

If X(k) is N-point DFT of a finite duration sequences x(n) then

$$DFT\left\{x((n-m))_N\right\} = X(k)e^{-\frac{j2\pi km}{N}}$$

Proof

IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{\frac{j2\pi kn}{N}} \quad ,0 \le n \le N-1$$

Put n=n-m

$$x(n-m) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{\frac{j2\pi k(n-m)}{N}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{\frac{j2\pi kn}{N}} e^{\frac{-j2\pi km}{N}}$$

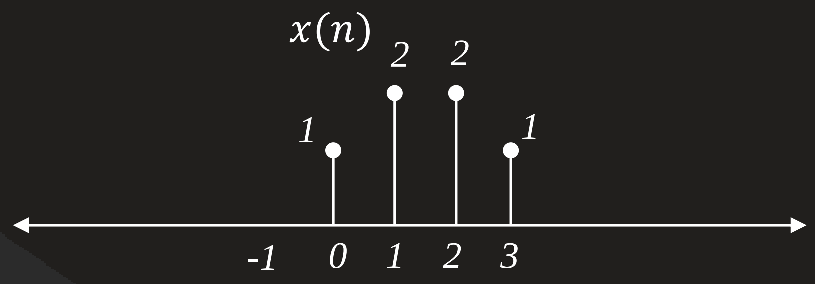
$$x(n-m) = x(n)e^{\frac{-j2\pi km}{N}}$$

Take DFT on both sides

$$DFT\{x(n-m)\} = X(k)e^{\frac{-j2\pi km}{N}}$$

Q) Consider a finite length sequences x(n) shown in figure. The five point DF of x(n) is denoted by X(k). Plot the sequences whose DFT is

$$Y(k) = e^{\frac{-4\pi k}{5}} X(k)$$



$$DFT\left\{x((n-m))_N\right\} = X(k)e^{-\frac{j2\pi km}{N}}$$

Solution

$$DFT\left\{x((n-2))_5\right\} = X(k)e^{-\frac{j2\pi k2}{5}}, n = 0,1,\dots,4$$

For n=0 → $y(0) = x((0-2))_5 = x(5+0-2) = x(3) = 1$

For n=1 → $y(1) = x((1-2))_5 = x(5+1-2) = x(4) = 0$

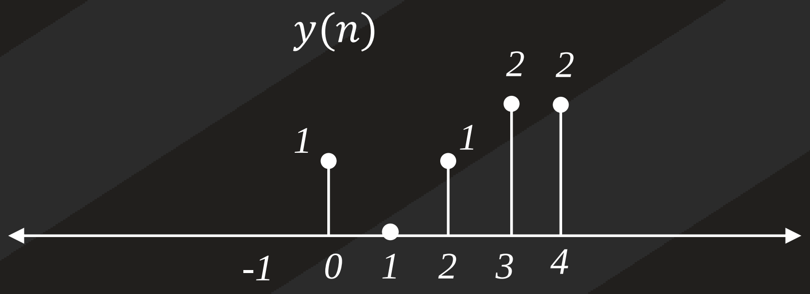
For n=2 → $y(2) = x((2-2))_5 = x(5+2-2) = x(5) = x(5-5) = x(0) = 1$

For n=3 → $y(3) = x((3-2))_5 = x(5+3-2) = x(6) = x(6-5) = x(1) = 2$

For n=4 → $y(4) = x((4-2))_5 = x(5+4-2) = x(7) = x(7-5) = x(2) = 2$

Exceeding the limit $0 \le n \le 4$

$$y(n) = \{1,0,1,2,2\}$$



Properties of Discrete Fourier Transform

Time reversal of the sequence

The time reversal of N-point sequence x(n) is attained by wrapping the sequence x(n) around the circle in clockwise direction.

$$x((-n))_N = DFT\{x(N-n)\} = X(N-k)$$

Circular frequency shift

If X(k) is N-point DFT of a finite duration sequences x(n) then

$$DFT\left[x(n)e^{\frac{j2\pi ln}{N}}\right] = X((k-l))_N$$

Proof

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi kn}{N}} \quad , 0 \le \mathrm{k} \le \mathrm{N}-1$$

Put k=k-l

$$X(k-l) = \sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi(k-l)n}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)ne^{-\frac{j2\pi kn}{N}}e^{\frac{j2\pi ln}{N}}$$

$$x(k-l) = X(k)e^{\frac{j2\pi ln}{N}}$$

Take DFT on both sides

$$X((k-l))_N = DFT\left\{x(n)e^{\frac{j2\pi ln}{N}}\right\}$$

Properties of Discrete Fourier Transform

Complex conjugate property

If X(k) is N-point DFT of a finite duration sequences x(n) then

$$DFT\{x^*(n)\} = X^*(N-k) = X^*((-k))_N$$

Proof

$$DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi kn}{N}}$$

$$DFT\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n)e^{-\frac{j2\pi kn}{N}}$$

$$DFT\{x(n)\} = \left[\sum_{n=0}^{N-1} x(n)e^{\frac{j2\pi kn}{N}}\right]^*$$

$$e^{-\frac{j2\pi nN}{N}} = e^{-j2\pi n} = 1$$

$$= \left[\sum_{n=0}^{N-1} x(n)e^{\frac{j2\pi kn}{N}} e^{-\frac{j2\pi nN}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)e^{-\frac{j2\pi n(N-k)}{N}}\right]^*$$

$$DFT\{x(n)\} = [X(N-k)]^*$$

Q) Let X(k) be a 14 point DFT of a length 14 real sequence x(n). The first 8 samples of X(k) are given by,

X(0) = 12,

X(1) = -1+3j ,

X(2) = 3+4j,

X(3) = 1-5j,

X(4) = -2+2j,

X(5) = 6+3j,

X(6) = -2-3j,

X(7) = 10. *Determine the remining samples*

$$DFT\{x(n)\} = [X(N-k)]^*$$

Solution

Given N=14

For n=8 ➔ $X(8) = X^*(N-k) = X^*(14-8) = X^*(6) = -2+3j$

For n=9 ➔ $X(9) = X^*(N-k) = X^*(14-9) = X^*(5) = 6-3j$

For n=10 ➔ $X(10) = X^*(N-k) = X^*(14-10) = X^*(4) = -2-2j$

For n=11 ➔ $X(11) = X^*(N-k) = X^*(14-11) = X^*(3) = 1+5j$

For n=12 ➔ $X(12) = X^*(N-k) = X^*(14-12) = X^*(2) = 3-4j$

For n=13 ➔ $X(13) = X^*(N-k) = X^*(14-13) = X^*(1) = -1-3j$

Linear Convolution

Consider a discrete sequence x(n) of length L and impulse sequence h(n) of length M, the equation for linear convolution is

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

Where length of y(n) is L+M-1

Let's discuss it with an example

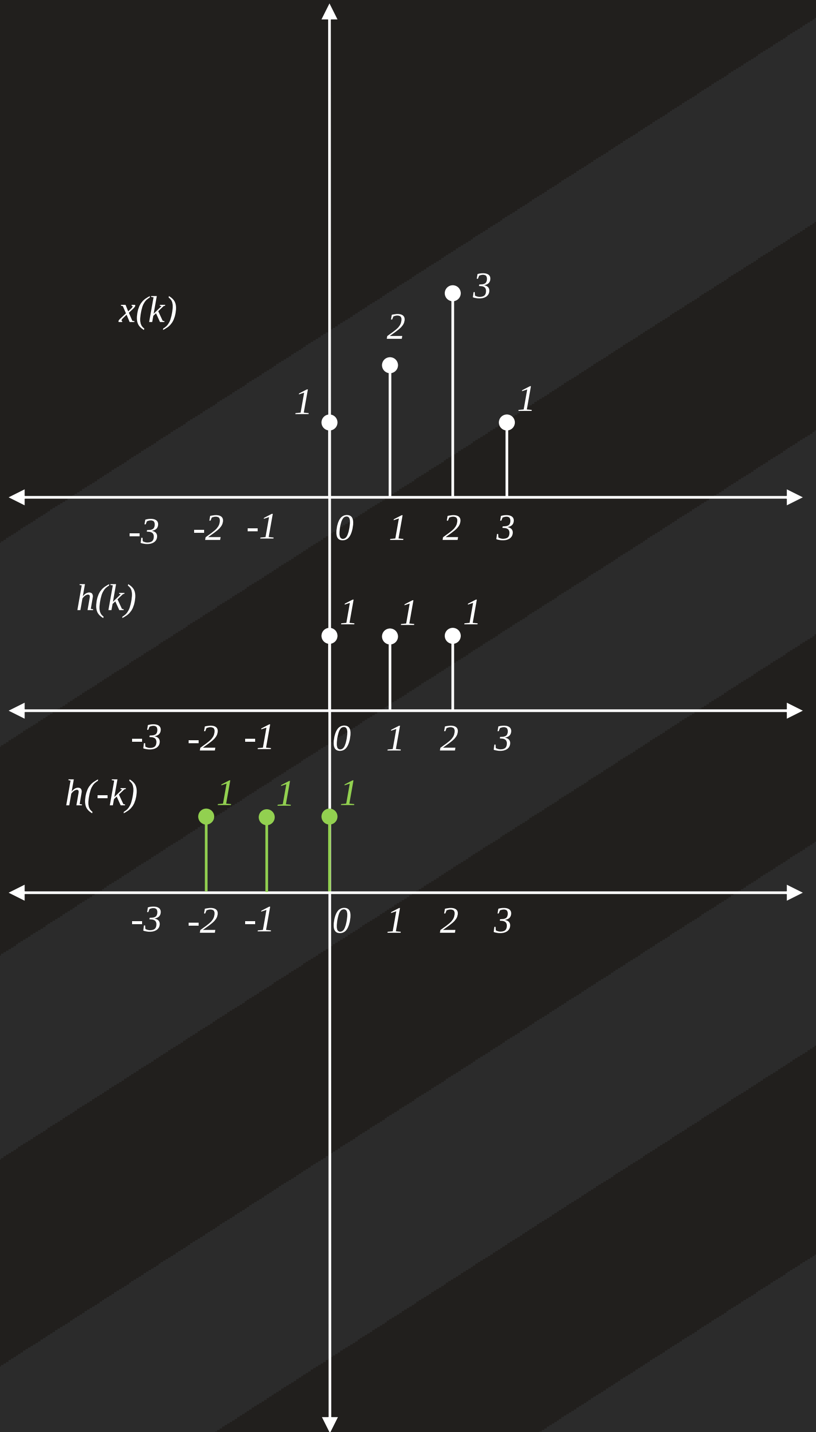
Q) Find the convolution of x(n) = {1,2,3,1}, h(n)={1,1,1,}

Solution

L = 4, M = 3

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$ *Of length → 4+3-1 = 6*

For n=0 ➔ $$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k) = (1.0) + (1.0) + (1.1) + (0.2) + (0.3) + (0.1) = 1$$



For n=1 ➔ $y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k) = \sum_{k=-\infty}^{\infty} x(k)h(-k+1)$

$= (1.0) + (1.1) + (1.2) + (0.3) + (0.1) = 3$

For n=2 ➔ $y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k) = \sum_{k=-\infty}^{\infty} x(k)h(-k+2)$

$= (1.1) + (1.2) + (1.3) + (0.1) = 6$

For n=3 ➔ $y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k) = \sum_{k=-\infty}^{\infty} x(k)h(-k+3)$

$= (0.1) + (1.2) + (1.3) + (1.1) = 6$

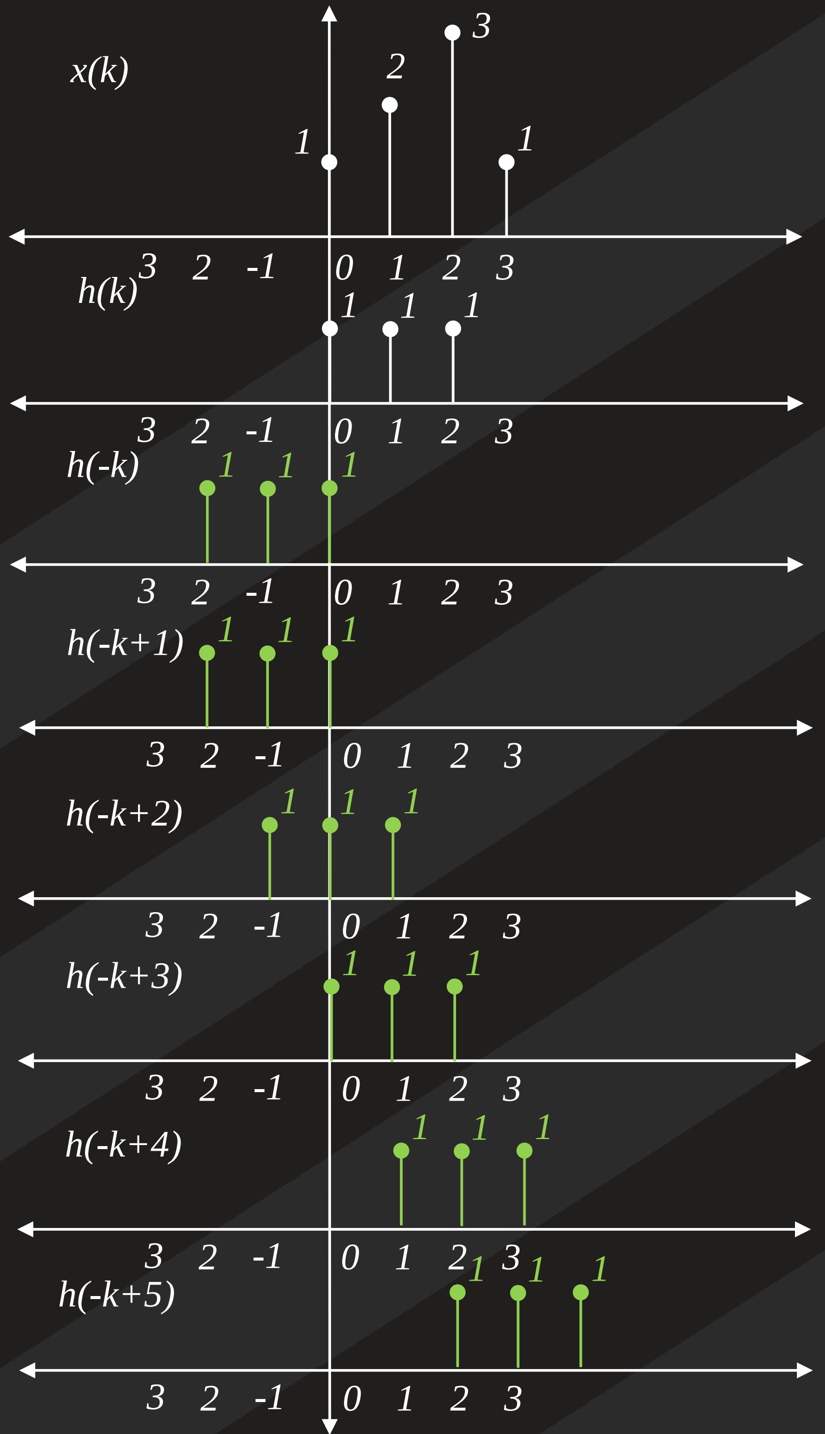
For n=4 ➔ $y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k) = \sum_{k=-\infty}^{\infty} x(k)h(-k+4)$

$= (0.1) + (0.2) + (1.3) + (1.1) = 4$

For n=5 ➔ $y(5) = \sum_{k=-\infty}^{\infty} x(k)h(5-k) = \sum_{k=-\infty}^{\infty} x(k)h(-k+5)$

$= (0.1) + (0.2) + (0.3) + (1.1) = 1$

$y(n) = \{1,3,6,6,4,1\}$



Q) Find the convolution of x(n) = {1,2,3,1}, h(n)={1,1,1,}

Solution

	1	1	1
1	1	1	1
2	2	2	2
3	3	3	3
1	1	1	1

$$y(n) = \{1,3,6,6,4,1\}$$

Circular Convolution

Consider two discrete sequence $x_1(n)$ & $x_2(n)$ of length N with DFTs $X_1(k)$, $X_2(k)$

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)x_2\big((n-m)\big)_N$$

$$x_3(n) = x_1(n) \odot x_2(n)$$

Also

$$DFT\{x_1(n) \odot x_2(n)\} = X_1(k).X_2(k)$$

Matrix method

Let's discuss it with an example

Q) Find the circular convolution of x(n) = {1,2,3,4}, h(n)={1,-1,1,}

Solution

L = 4, M = 3

Since lengths are not same we do zero-padding

$$h(n) = \{1,-1,1,0\}$$

$$\begin{bmatrix} 1 & & 3 & 2 \\ 2 & & 4 & 3 \\ 3 & & 1 & 4 \\ 4 & & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (1.1)+(4.-1)+(3.1)+(2.0) = 0 \\ (2.1)+(1.-1)+(4.1)+(3.0) = 5 \\ (3.1)+(2.-1)+(1.1)+(4.0) = 2 \\ (4.1)+(3.-1)+(2.1)+(2.0) = 3 \end{bmatrix}$$

$$y(n) = \{0,5,2,3\}$$

Circular Convolution

Concentric Circle method / Stockholm's Method

Let's discuss it with an example

Q) Find the circular convolution of x(n) = {1,2,3,4}, h(n)={1,-1,1,}

Solution

L = 4, M = 3

Since lengths are not same we do zero-padding

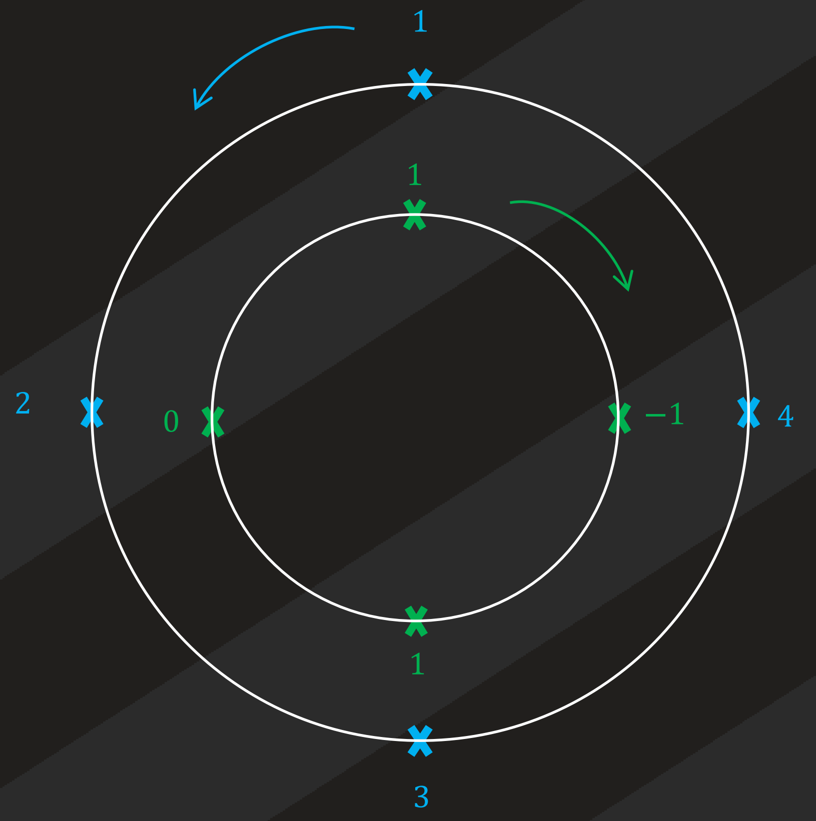
$$h(n) = \{1, -1, 1, 0\}$$

For n=0 ➔ $y(0) = (1.1) + (2.0) + (3.1) + (4.-1) \quad = 0$

For n=1 ➔ $y(1) = (1.-1) + (2.1) + (3.0) + (4.1) \quad = 5$

For n=2 ➔ $y(2) = (1.1) + (2.-1) + (3.1) + (4.0) \quad = 2$

For n=3 ➔ $y(3) = (1.0) + (2.1) + (3.-1) + (4.1) \quad = 3$



$$y(n) = \{0,5,2,3\}$$

Linear convolution using circular convolution

Let there are two sequence x(n) with L and h(n) with length M. in linear convolution the length of output in L+M-1. In circular convolution the length of the both input is L=M

Let's discuss with an example

Q) Find the convolution of the sequences x(n) = {1,2,3,1}, h(n)={1,1,1,}

First we have to make the length of the x(n) and h(n) by adding zeros

$x(n) = \{1,2,3,1,0,0\}$ *(M-1 zeros)*

$h(n) = \{1,1,1,0,0,0\}$ *(L-1 zeros)*

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 3 & 2 \\ 2 & 1 & 0 & 0 & 1 & 3 \\ 3 & 2 & 1 & 0 & 0 & 1 \\ 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} (1.1)+(0.1)+(0.1)+(1.0)+(3.0)+(2.0) = 1 \\ (2.1)+(1.1)+(0.1)+(0.0)+(1.0)+(3.0) = 3 \\ (3.1)+(2.1)+(1.1)+(0.0)+(0.0)+(1.0) = 6 \\ (1.1)+(3.1)+(2.1)+(1.0)+(0.0)+(0.0) = 6 \\ (0.1)+(1.1)+(3.1)+(2.0)+(1.0)+(0.0) = 4 \\ (0.1)+(0.1)+(1.1)+(3.0)+(2.0)+(1.0) = 1 \end{bmatrix}$$

$y(n) = \{1,3,6,6,4,1\}$

Filtering of long duration sequences

1) Overlap – save method

Let's consider an input sequence x(n) of length L_s *and response h(n)of length M, the steps to follow overlap – save method is*

Step 1 : input x(n) is divided into length L (L≥M)

Step 2 : Calculate the length N=L+M-1

Step 3 : Add M-1 zeros to the start to first segment, each segment (length = L) has its first M-1 points coming from previous segment, making each of length N

Step 4 : Make impulse response to length N by adding zeros

Step 5 ; Find the circular convolution of each new segments with new h(n)

Step 6 : Linearly combine each results and take sequence of length L_s*+M-1 from that by discarding/removing first M-1 points*

1) Overlap – save method

Q) Find the convolution of the sequences x(n) = {3,-1,0,1,3,2,0,1,2,1} and h(n) ={1,1,1}

Solution

Given , Ls = 10 & M=3 *Lets guess the value of L =3* $(L \geq M)$

Step 1 : input x(n) is divided into length L

$$x_1(n) = \{3, -1, 0\}$$

$$x_2(n) = \{1, 3, 2\}$$

$$x_3(n) = \{0, 1, 2\}$$

$$x_4(n) = \{1, 0, 0\}$$

Step 2 : Calculate the length N=L+M-1

$$N = L + M - 1 \quad = 3 + 3 - 1 = 5$$

1) Overlap – save method

Step 3 : Add M-1 zeros to the start to first segment, each segment (length = L) has its first M-1 points coming from previous segment, making each of length N

$$x_1(n) = \{0,0,3,-1,0\}$$

$$x_2(n) = \{-1,0,1,3,2\}$$

$$x_3(n) = \{3,2,0,1,2\}$$

$$x_4(n) = \{1,2,1,0,0\}$$

M-1 = 3-1 = 2

$$x_1(n) = \{3,-1,0\}$$

$$x_2(n) = \{1,3,2\}$$

$$x_3(n) = \{0,1,2\}$$

$$x_4(n) = \{1,0,0\}$$

Step 4 : Make impulse response to length N by adding zeros

$$h(n) = \{1,1,1,0,0\}$$

1) Overlap – save method

Step 5 ; Find the circular convolution of each new segments with new h(n)

$$y_1(n) = x_1(n) \odot h(n) = \{0,0,3,-1,0\} \odot \{1,1,1,0,0\} = \{-1,0,3,2,2\}$$

$$y_2(n) = x_2(n) \odot h(n) = \{-1,0,1,3,2\} \odot \{1,1,1,0,0\} = \{4,1,0,4,6\}$$

$$y_3(n) = x_3(n) \odot h(n) = \{3,2,0,1,2\} \odot \{1,1,1,0,0\} = \{6,7,5,3,3\}$$

$$y_4(n) = x_4(n) \odot h(n) = \{1,2,1,0,0\} \odot \{1,1,1,0,0\} = \{1,3,4,3,1\}$$

$$\begin{bmatrix} 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 3 \\ 3 & 0 & 0 & 0 & -1 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Step 6 : Linearly combine each results and take sequence of length L_s+M-1 from that by discarding/removing first M-1 points

M-1 = 3-1 = 2

Check whether length of y(n) is L_s+M-1 , if yes discard the higher sequences

$$y(n) = \{3,2,2,0,4,6,5,3,3,4,3,1\}$$

L_s+M-1 = 10+3-1 = 12

1) Overlap – save method

Q) Find the convolution of the sequences x(n) = {1,2,-1,2,3,-2,-3,-1,1,1,2,-1} and h(n) ={1,2} using overlap-save method

Solution

Given , Ls = 12 & M=2 *Lets guess the value of L =3* $(L \geq M)$

Step 1 : input x(n) is divided into length L

$$x_1(n) = \{1, 2, -1\}$$

$$x_2(n) = \{2,3, -2\}$$

$$x_3(n) = \{-3, -1,1\}$$

$$x_4(n) = \{1,2, -1\}$$

Step 2 : Calculate the length N=L+M-1

$$N = L + M - 1 \quad = 3 + 2 - 1 = 4$$

1) Overlap – save method

Step 3 : Add M-1 zeros to the start to first segment, each segment (length = L) has its first M-1 points coming from previous segment, making each of length N

$$x_1(n) = \{0,1,2,-1\}$$

$$x_2(n) = \{-1,2,3,-2\}$$

$$x_3(n) = \{-2,-3,-1,1\}$$

$$x_4(n) = \{1,1,2,-1\}$$

$$x_5(n) = \{-1,0,0,0\}$$

M-1 = 2-1 = 1

$$x_1(n) = \{1,2,-1\}$$

$$x_2(n) = \{2,3,-2\}$$

$$x_3(n) = \{-3,-1,1\}$$

$$x_4(n) = \{1,2,-1\}$$

Step 4 : Make impulse response to length N by adding zeros

$$h(n) = \{1,2,0,0\}$$

1) Overlap – save method

Step 5 ; Find the circular convolution of each new segments with new h(n)

$$y_1(n) = x_1(n) \odot h(n) = \{0,1,2,-1\} \odot \{1,2,0,0\} = \{-2, 1, 4, 3\}$$

$$y_2(n) = x_2(n) \odot h(n) = \{-1,2,3,-2\} \odot \{1,2,0,0\} = \{-5, 0, 7, 4\}$$

$$y_3(n) = x_3(n) \odot h(n) = \{-2,-3,-1,1\} \odot \{1,2,0,0\} = \{0, -7, -7, -1\}$$

$$y_4(n) = x_4(n) \odot h(n) = \{1, 1, 2, -1\} \odot \{1,2,0,0\} = \{-1, 3, 4, 3\}$$

$$y_5(n) = x_5(n) \odot h(n) = \{-1, 0,0,0\} \odot \{1,2,0,0\} = \{-1, -2, 0, 0\}$$

$$\begin{bmatrix} 0 & -1 & 2 & 1 \\ 1 & 0 & -1 & 2 \\ 2 & 1 & 0 & -1 \\ -1 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Step 6 : Linearly combine each results and take sequence of length L_s+M-1 from that by discarding/removing first M-1 points

$$y(n) = \{1,4,3,0,7,4,-7,-7,-1,3,4,3,-2,0,0\}$$

Check whether length of y(n) is L_s+M-1 , if yes discard the higher sequences

$L_s+M-1 = 12+2-1 = 13$

$$y(n) = \{1,4,3,0,7,4,-7,-7,-1,3,4,3,-2\}$$

Filtering of long duration sequences

2) Overlap – add method

Let's consider an input sequence x(n) of length L_1 and response h(n) of length M, the steps to follow overlap – save method is

Step 1 : input x(n) is divided into length L ($L \geq M$)

Step 2 : Calculate the length $N=L+M-1$

Step 3 : Add M-1 zeros on each segment (length = L) of x(n)

Step 4 : Make impulse response to length N by adding zeros

Step 5 ; Find the circular convolution of each new segments with new h(n)

Step 6 : Add last and first M-1 points of each segments, discard/remove excess point than L_1+M-1

1) Overlap – add method

Q) Find the convolution of the sequences x(n) = {3,-1,0,1,3,2,0,1,2,1} and h(n) ={1,1,1}

Solution

Given , L_1 = 10 & M=3 Lets guess the value of L =3 $(L \leq M)$

Step 1: input x(n) is divided into length L (L≥M)

$$x_1(n) = \{3,-1,0\}$$

$$x_2(n) = \{1,3,2\}$$

$$x_3(n) = \{0,1,2\}$$

$$x_4(n) = \{1,0,0\}$$

Step 2 : Calculate the length N=L+M-1

$$N = L + M - 1 \quad = 3 + 3 - 1 = 5$$

1) Overlap – add method

Step 3 : Add M-1 zeros on each segment (length = L) of x(n)

$x_1(n) = \{3, -1, 0, 0, 0\}$

$x_2(n) = \{1, 3, 2, 0, 0\}$

$x_3(n) = \{0, 1, 2, 0, 0\}$

$x_4(n) = \{1, 0, 0, 0, 0\}$

M-1 = 3-1 = 2

$x_1(n) = \{3, -1, 0\}$

$x_2(n) = \{1, 3, 2\}$

$x_3(n) = \{0, 1, 2\}$

$x_4(n) = \{1, 0, 0\}$

Step 4 : Make impulse response to length N by adding zeros

$h(n) = \{1,1,1,0,0\}$

1) Overlap – add method

Step 5 ; Find the circular convolution of each new segments with new h(n)

$$y_1(n) = x_1(n) \odot h(n) = \{3, -1, 0, 0, 0,\} \odot \{1,1,1,0,0\} = \{3,2,2,-1,0\}$$

$$y_2(n) = x_2(n) \odot h(n) = \{1, 3, 2, 0, 0\} \odot \{1,1,1,0,0\} = \{1,4,6,5,2\}$$

$$y_3(n) = x_3(n) \odot h(n) = \{0, 1, 2, 0, 0\} \odot \{1,1,1,0,0\} = \{0,1,3,3,2\}$$

$$y_4(n) = x_4(n) \odot h(n) = \{1, 0, 0, 0, 0\} \odot \{1,1,1,0,0\} = \{1,1,1,0,0\}$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 & -1 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Step 6 : Add last and first M-1 points of each segments, discard/remove excess point than L_1+M-1

$$\{3, 2, 2, -1, 0\}$$
$$\{1, 4, 6, 5, 2\}$$
$$\{0, 1, 3, 3, 2\}$$
$$\{1, 1, 1, 0, 0\}$$

$$\{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1, 0, 0\}$$

Check whether length of y(n) is L_1+M-1 , if yes discard the higher sequences

$$L_1 + M - 1 = 10 + 3 - 1 = 12$$

$$y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$$

1) Overlap – add method

Q) Find the convolution of the sequences x(n) = {1,2,-1,2,3,-2,-3,-1,1,1,2,-1} and h(n) ={1,2} using overlap-add method

Solution

Given , Ls = 12 & M=2 Lets guess the value of L =3 $(L \geq M)$

Step 1 : input x(n) is divided into length L

$$x_1(n) = \{1, 2, -1\}$$

$$x_2(n) = \{2,3, -2\}$$

$$x_3(n) = \{-3, -1,1\}$$

$$x_4(n) = \{1,2, -1\}$$

Step 2 : Calculate the length N=L+M-1

$$N = L + M - 1 \quad = 3 + 2 - 1 = 4$$

1) Overlap – save method

Step 3 : Add M-1 zeros on each segment (length = L) of x(n)

$x_1(n) = \{1, 2, -1, 0\}$

$x_2(n) = \{2,3,-2,0\}$

$x_3(n) = \{-3,-1,1,0\}$

$x_4(n) = \{1, 2, -1,0\}$

M-1 = 2-1 = 1

$x_1(n) = \{1, 2, -1\}$

$x_2(n) = \{2,3,-2\}$

$x_3(n) = \{-3,-1,1\}$

$x_4(n) = \{1,2,-1\}$

Step 4 : Make impulse response to length N by adding zeros

$h(n) = \{1, 2 , 0 , 0 \}$

1) Overlap – save method

Step 5 ; Find the circular convolution of each new segments with new h(n)

$$y_1(n) = x_1(n) \odot h(n) = \{1,2,-1,0\} \odot \{1,2,0,0\} = \{1,4,3,2\}$$

$$y_2(n) = x_2(n) \odot h(n) = \{2,3,-2,0\} \odot \{1,2,0,0\} = \{2,7,4,-4\}$$

$$y_3(n) = x_3(n) \odot h(n) = \{-3,-1,1,0\} \odot \{1,2,0,0\} = \{-3,-7,-1,2\}$$

$$y_4(n) = x_4(n) \odot h(n) = \{1,2,-1,0\} \odot \{1,2,0,0\} = \{1,4,3,-2\}$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 0 & -1 \\ -1 & 2 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

Step 6 : Add last and first M-1 points of each segments, discard/remove excess point than L1+M-1

$$\{1,4,3,2\}$$
$$\{-2,7,4,-4\}$$
$$\{-3,-7,-1,2\}$$
$$\{1,4,3,-2\}$$

$$\{1,4,3,0,7,4,-7,-7,-1,3,4,3,-2,\}$$

Check whether length of y(n) is L_s+M-1 , if yes discard the higher sequences

$L_s+M-1 = 12+2-1 = 13$

$$y(n) = \{1,4,3,0,7,4,-7,-7,-1,3,4,3,-2\}$$

Fast Fourier Transform (FFT)

Lets calculate the DFT of a sequence with N=4 and k=1

$$X(1) = \sum_{n=0}^{3} x(n)W_4^n$$

$$= x(0)W_4^0 + x(1)W_4^1 + x(2)W_4^2 + x(3)W_4^3$$

For a single value of k we have

4 → Multiplication

3 → Addition

For example N=1024

DFT method

$$Complex\ multiplication = N^2$$
$$= 1024^2$$
$$= 1048576$$

$$Complex\ Addition = N(N-1)$$
$$= 1024(1024-1)$$
$$= 1047552$$

MODULE 2 Fast Fourier Transform (FFT)

- *DFT takes more time and resources*
- *Not much efficient*
- *Much complex*

- *So we come into a new algorithm to make the calculations fast known as fast Fourier Transform (FFT)*
- *It is a highly efficient procedure for computing the DFT of a sequence for computing the DFT of a finite sequence and require less number of computation than that of direct evaluation of DFT*
- *FFT is based on decomposition and breaking the transform into smaller transform and combine them to get total transform*
- *FFT make use of the symmetry and periodicity property of twiddle factor*

FFT method

For example N=1024

$$Complex\ multiplication = \frac{N}{2}\log_2 N$$

$$= \frac{1024}{2}\log_2 1024$$

$$= 5120$$

$$Complex\ Addition = N\log_2 N$$

$$= 1024\log_2 1024$$

$$= 1024$$

Fast Fourier Transform (FFT)

Let us recollect the twiddle factor

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \quad ,0 \le \mathrm{k} \le \mathrm{N}-1$$

$$W_N = e^{-\frac{j2\pi}{N}}$$

$$W_N^k = e^{\left(-\frac{j2\pi}{N}\right)k}$$

For N=4 & k=0

$$W_4^0 = e^{\left(-\frac{j2\pi}{4}\right)0} = 1$$

For N=4 & k=1

$$W_4^1 = e^{\left(-\frac{j2\pi}{4}\right).1} = e^{-\frac{j\pi}{2}} = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right)$$

$$W_4^1 = -j$$

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

For N=8 & k=0

$$W_8^0 = e^{\left(-\frac{j2\pi}{8}\right)0} = 1$$

For N=8 & k=1

$$W_8^1 = e^{\left(-\frac{j2\pi}{8}\right).1} = e^{\frac{-j\pi}{4}}$$

$$= \cos\left(\frac{\pi}{4}\right) - j\sin\left(\frac{\pi}{4}\right)$$

$$W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = 0.7071 - j0.7071$$

For N=8 & k=2

$$W_8^2 = e^{\left(-\frac{j2\pi}{8}\right).2} = e^{\frac{-j\pi}{2}}$$

$$= \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right)$$

$$W_8^2 = -j$$

For N=8 & k=3

$$W_8^3 = e^{\left(-\frac{j2\pi}{8}\right).3} = e^{\frac{-j3\pi}{4}}$$

$$= \cos\left(\frac{3\pi}{4}\right) - j\sin\left(\frac{3\pi}{4}\right)$$

$$W_8^3 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^3 = -0.7071 - j0.7071$$

Fast Fourier Transform (FFT)

Decimation in Time (DIT)

- Also known as Radix DIT FFT algorithm
- The number of output points N can be expressed as a power of 2 ($N=2^M$)

Let x(n) is an N-point sequence and we are dividing it into two (even $x_e(n)$ & odd $x_o(n)$) parts

$$x_e(n) = x(2n) \qquad x_o(n) = x(2n+1)$$

We know DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{(2n+1)k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nk} + W_N^k \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{2nk}$$

$$W_N^2 = e^{\left(-\frac{j2\pi}{N}\right)2} = e^{\left(-\frac{j2\pi}{\frac{N}{2}}\right)} = W_{\frac{N}{2}}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x_e(n) W_{\frac{N}{2}}^{nk} + W_N^k \sum_{n=0}^{\frac{N}{2}-1} x_o(n) W_{\frac{N}{2}}^{nk}$$

$$X(k) = X_e(k) + W_N^k X_o(k) \qquad \text{For } k < N/2$$

From symmetry property : $W_N^{k+\frac{N}{2}} = -W_N^k$

Then

$$X(k) = X_e\left(k-\frac{N}{2}\right) - W_N^{k-\frac{N}{2}} X_o\left(k-\frac{N}{2}\right) \qquad \text{For } k > N/2$$

Decimation in Time (DIT)

$$X(k) = X_e(k) + W_N^k X_o(k)$$

$$X(k) = X_e\left(k - \frac{N}{2}\right) - W_N^{k-\frac{N}{2}} X_o\left(k - \frac{N}{2}\right)$$

Example : For N = 8

even	odd
$x_e(0) = x(0)$	$x_o(0) = x(1)$
$x_e(1) = x(2)$	$x_0(1) = x(3)$
$x_e(2) = x(4)$	$x_o(2) = x(5)$
$x_e(3) = x(6)$	$x_o(3) = x(7)$

$$X(k) = x_e(k) + W_8^k x_o(k), \qquad for\ 0 \le k \le 3$$

$$X(k) = x_e(k-4) - W_8^{k-4} x_o(k-4), \qquad for\ 4 \le k \le 7$$

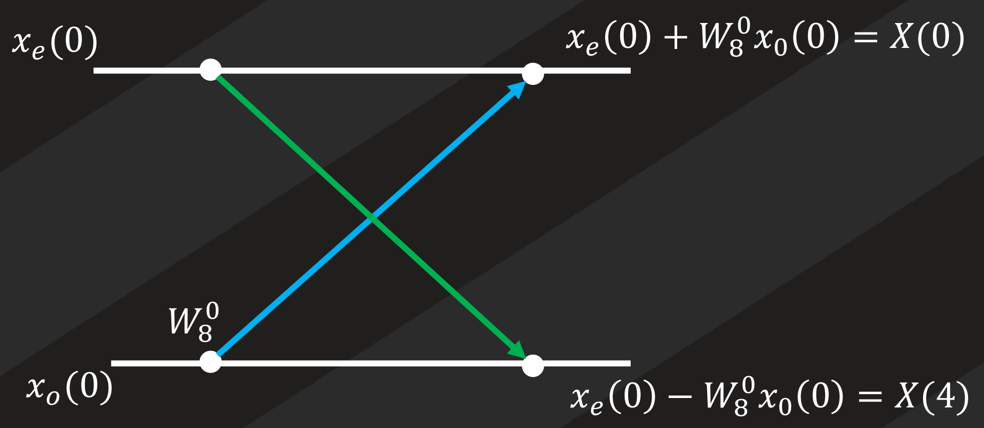
$$X(0) = x_e(0) + W_8^0 x_o(0) \qquad X(4) = x_e(0) - W_8^0 x_o(0)$$

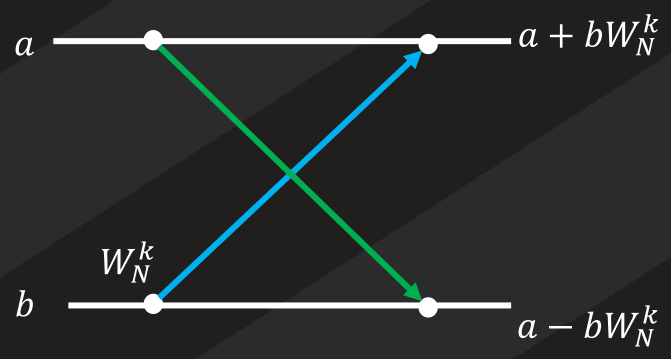
$$X(1) = x_e(1) + W_8^1 x_o(1) \qquad X(5) = x_e(1) - W_8^1 x_o(1)$$

$$X(2) = x_e(2) + W_8^2 x_o(2) \qquad X(6) = x_e(2) - W_8^2 x_o(2)$$

$$X(3) = x_e(3) + W_8^3 x_o(3) \qquad X(7) = x_e(3) - W_8^3 x_o(3)$$

This operation can be represented by a butterfly diagram





Decimation in Time (DIT)

Steps to follow

Step 1 : Find the number of input samples (N)

Step 2 : Bir reversal

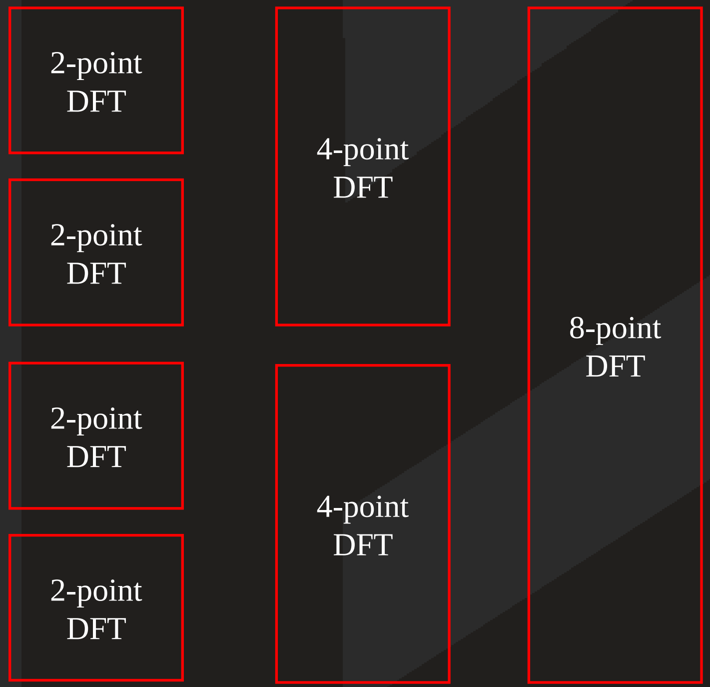
Step 3 : Calculate the number of stages $(M = \log_2 N)$

Step 4 : Calculate the number of max butterflies in stage (N/2)

Step 5 : Calculate the twiddle factor

Step 6 : Evaluate the N point DFT using butterfly diagram

Step7 : The DFT output is in normal order



Input	Binary	Bit-reversed	Revised samples
x(0)	000	000	x(0)
x(1)	001	100	x(4)
x(2)	010	010	x(2)
x(3)	011	110	x(6)
x(4)	100	001	x(1)
x(5)	101	101	x(5)
x(6)	110	011	x(3)
x(7)	111	111	x(7)

Decimation in Time (DIT)

Q) Find the DFT of a sequence x(n) = {0, 1, 2, 3} using DIT algorithm

Solution

Step 1 : Find the number of input samples (N)

N=4

Step 2 : Bit reversal

Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)

Step 3 : Calculate the number of stages $(M = \log_2 N)$

$$M = \log_2 N \quad = \log_2 4$$

$$M = 2$$

Step 4 : Calculate the number of max butterflies in stage

$$\frac{N}{2} = \frac{4}{2} = 2$$

Decimation in Time (DIT)

Step 5 : Calculate the twiddle factor

$$k = \frac{Nt}{2^M} \qquad t = 0, 1, 2, \ldots 2^{M-1} - 1$$

Stage =1 $(M = 1)$

$t = 0$

$$k = \frac{Nt}{2^M} = \frac{4.0}{2^1} = 0$$

W_4^0

$$W_4^0 = e^{\left(-\frac{j2\pi}{4}\right)0} = 1$$

Stage =2 $(M = 2)$

$t = 0,1$

for t=0

$$k = \frac{4.0}{2^2} = 0$$

for t=1

$$k = \frac{4.1}{2^2} = 1$$

$W_4^0 \qquad W_4^1$

$$W_4^1 = e^{\left(-\frac{j2\pi}{4}\right).1} = e^{-\frac{j\pi}{2}} = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right)$$

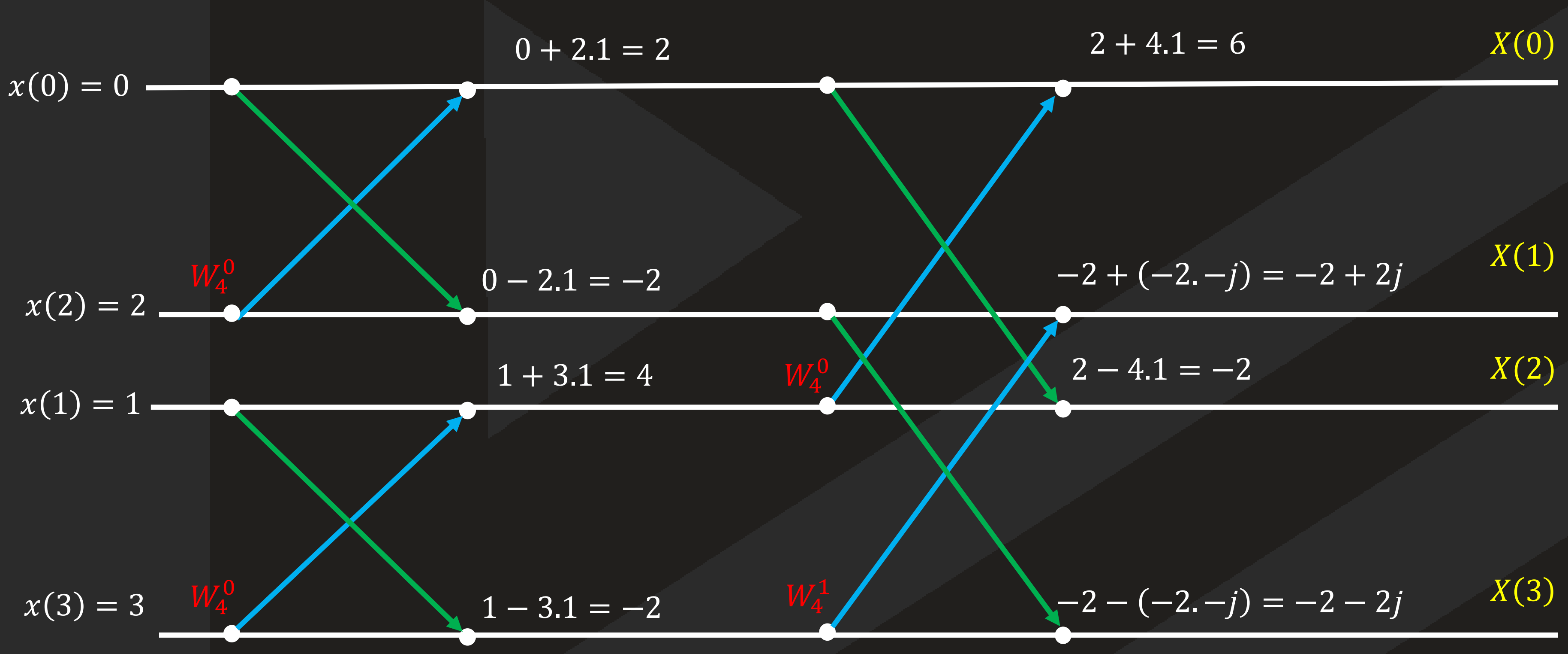
$$W_4^1 = -j$$

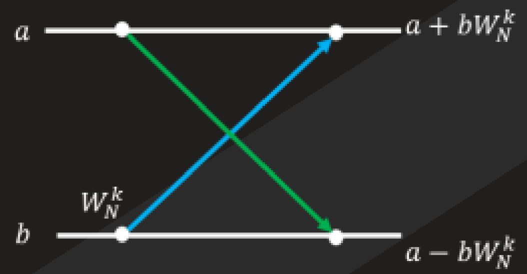
Decimation in Time (DIT)

Step 6 : Evaluate the N point DFT using butterfly diagram

$x(n) = \{0, 1, 2, 3\}$

$W_4^0 = 1 \quad W_4^1 = -j$





Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)

Step7 : The DFT output is in normal order

$X(k) = \{6, -2 + 2j, -2 \;\; -2 - 2j\}$

Decimation in Time (DIT)

Q) Find the DFT of a sequence x(n) = {1, 2, 3, 4, 4, 3, 2, 1 } using DIT algorithm

Solution

Step 1 : Find the number of input samples (N)

N=8

Step 2 : Bit reversal

Input	Binary	Bit-reversed	Revised samples
x(0)	000	000	x(0)
x(1)	001	100	x(4)
x(2)	010	010	x(2)
x(3)	011	110	x(6)
x(4)	100	001	x(1)
x(5)	101	101	x(5)
x(6)	110	011	x(3)
x(7)	111	111	x(7)

Step 3 : Calculate the number of stages ($M = \log_2 N$)

$$M = \log_2 N \quad = \log_2 8$$

$$M = 3$$

Step 4 : Calculate the number of max butterflies in stage

$$\frac{N}{2} = \frac{8}{2} = 4$$

Decimation in Time (DIT)

Step 5 : Calculate the twiddle factor

$$k = \frac{Nt}{2^M} \qquad t = 0, 1, 2, \ldots 2^{M-1} - 1$$

Stage =1 $(M = 1)$

$t = 0$

$$k = \frac{Nt}{2^M} = \frac{8.0}{2^1} = 0$$

W_8^0

$$W_8^0 = e^{\left(-\frac{j2\pi}{8}\right)0} = 1$$

Stage =2 $(M = 2)$

$t = 0,1$

for t=0

$$k = \frac{8.0}{2^2} = 0 \qquad W_8^0$$

for t=1

$$k = \frac{8.1}{2^2} = 2 \qquad W_8^2$$

$$W_8^2 = -j$$

$$W_8^3 = e^{\left(-\frac{j2\pi}{8}\right).3} = e^{\frac{-j3\pi}{4}}$$

$$= \cos\left(\frac{3\pi}{4}\right) - j\sin\left(\frac{3\pi}{4}\right)$$

$$W_8^3 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^3 = -0.7071 - j0.7071$$

Stage =3 $(M = 3)$

$$W_8^2 = e^{\left(-\frac{j2\pi}{8}\right).2}$$

$$= e^{\frac{-j\pi}{2}}$$

$$= \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right)$$

$$W_8^2 = -j$$

$$W_8^1 = e^{\left(-\frac{j2\pi}{8}\right).1} = e^{\frac{-j\pi}{4}}$$

$$= \cos\left(\frac{\pi}{4}\right) - j\sin\left(\frac{\pi}{4}\right)$$

$$W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = 0.7071 - j0.7071$$

$t = 0,1,2,3$

for t=0

$$k = \frac{8.0}{2^3} = 0 \qquad W_8^0 \qquad W_8^0 = 1$$

for t=1

$$k = \frac{8.1}{2^3} = 1 \qquad W_8^1 \qquad W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

for t=2

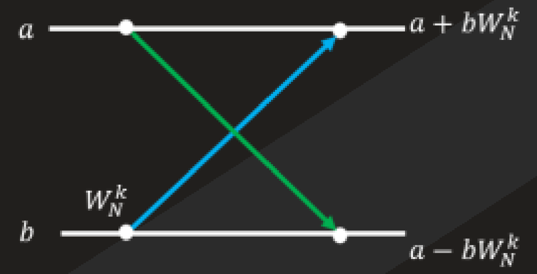
$$k = \frac{8.2}{2^3} = 2 \qquad W_8^2 \qquad W_8^2 = -j$$

for t=3

$$k = \frac{8.3}{2^3} = 3 \qquad W_8^3 \qquad W_8^2 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

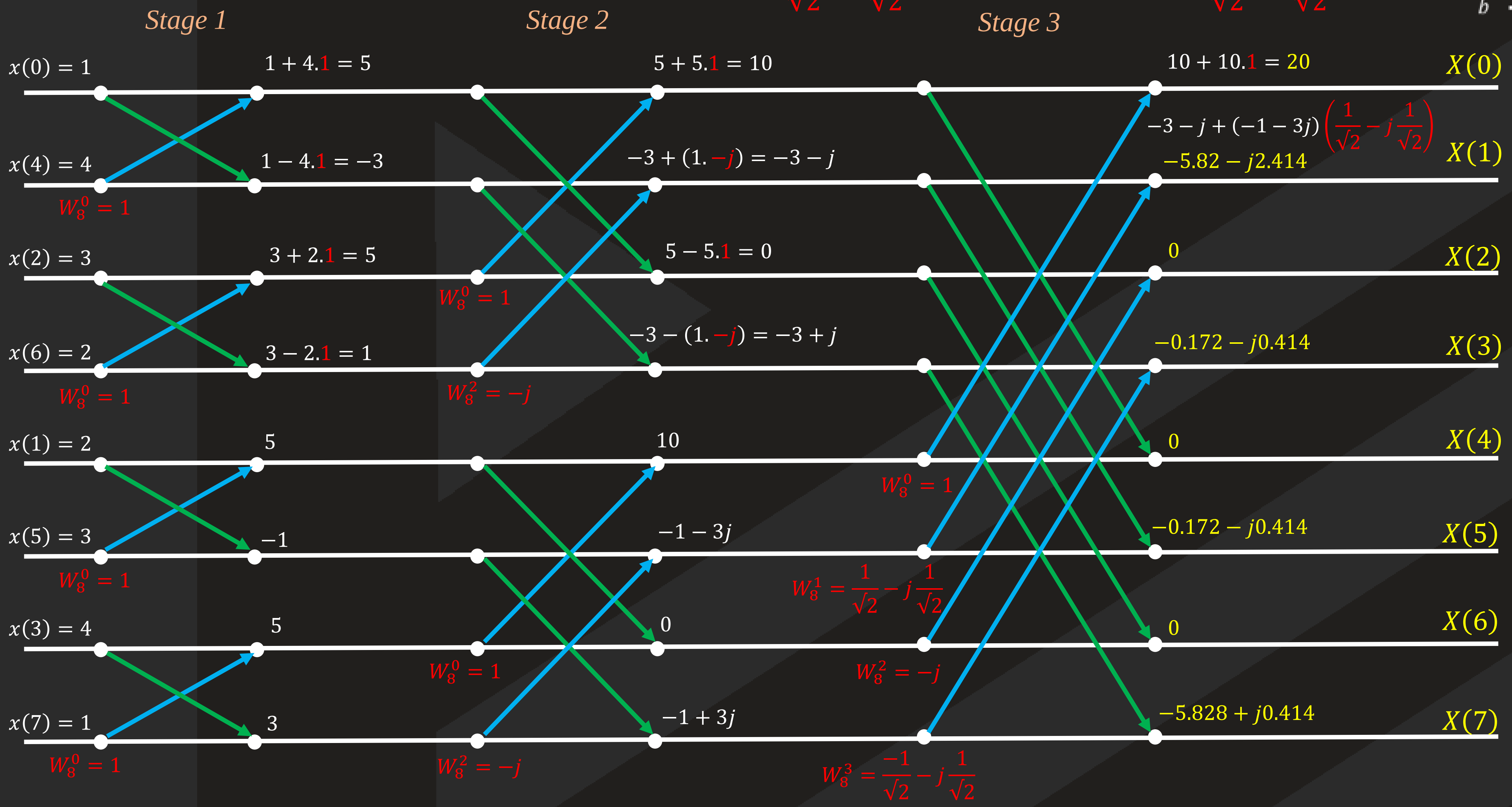
Decimation in Time (DIT)

Step 6 : Evaluate the N point DFT using butterfly diagram



$x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$

$W_8^0 = 1$ $\quad W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$ $\quad W_8^2 = -j$ $\quad W_8^3 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$



Input	Revised samples
x(0)	x(0)
x(1)	x(4)
x(2)	x(2)
x(3)	x(6)
x(4)	x(1)
x(5)	x(5)
x(6)	x(3)
x(7)	x(7)

Step7 : The DFT output is in normal order

$X(k) = \{20, -5.82 - j2.414, 0, -0.172 - j0.414, 0, -0.172 - j0.414, 0, -5.828 + j0.414\}$

Fast Fourier Transform (FFT)

$$W_N^{\frac{Nk}{2}} = e^{\left(\frac{-j2\pi}{N}\right)\frac{Nk}{2}} = e^{-j\pi k}$$

Decimation in Frequency (DIF)

- Based on the decomposition of the DFT computation by forming smaller and smaller sub sequences
- In DIF the output sequence X(k) is divided into smaller and smaller sub sequences

Let x(n) is an N-point sequence and we are dividing it into two parts

$$x_1(n) = x(n) \qquad x_2(n) = x\left(n + \frac{N}{2}\right)$$

$$n = 0,1,2,\ldots\frac{N}{2} - 1 \qquad n = 0,1,2,\ldots\frac{N}{2} - 1$$

We know DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n)W_N^{nk} + \sum_{n=0}^{\frac{N}{2}-1} x_2(n)W_N^{\left(n+\frac{N}{2}\right)k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n)W_N^{nk} + W_N^{\frac{NK}{2}} \sum_{n=0}^{\frac{N}{2}-1} x_2(n)W_N^{nk}$$

When k is even $e^{-j\pi k} = 1$

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} x_1(n)W_N^{nk} + e^{-j\pi k}\sum_{n=0}^{\frac{N}{2}-1} x_2(n)W_N^{nk} = \sum_{n=0}^{\frac{N}{2}-1}\left(x_1(n) + x_2(n)\right)W_N^{2nk}$$

$$W_N^{2nk} = W_{\frac{N}{2}}^{k}$$

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1}\left(x_1(n) + x_2(n)\right)W_{\frac{N}{2}}^{nk}$$

In the above equation in the N/2 –point DFT of N/2 sequences is obtained by adding the first half and last half of the input sequences

When k is odd $e^{-j\pi k} = -1$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1}\left(x_1(n) - x_2(n)\right)W_N^{(2k+1)n}$$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1}\left(x_1(n) - x_2(n)\right)W_{\frac{N}{2}}^{nk}\, W_N^{n}$$

In the above equation in the N/2 –point DFT of N/2 sequences is obtained by subtracting the second half of the input from the first half and then multiplying the result with W_N^k

Decimation in Frequency (DIF)

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} (x_1(n) + x_2(n)) W_{\frac{N}{2}}^{nk}$$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} (x_1(n) - x_2(n)) W_{\frac{N}{2}}^{nk} W_N^n$$

Example : For N = 8

$x_1(n)$	$x_2(n)$
$x_1(0) = x(0)$	$x_2(0) = x(4)$
$x_1(1) = x(1)$	$x_2(1) = x(5)$
$x_1(2) = x(2)$	$x_2(2) = x(6)$
$x_1(3) = x(3)$	$x_2(3) = x(7)$

$$X(2k) = x_1(k) + x_2(k), \quad for\ 0 \le k \le 3$$

$$X(2k+1) = [x_1(k) - x_2(k)]W_8^k, \quad for\ 4 \le k \le 7$$

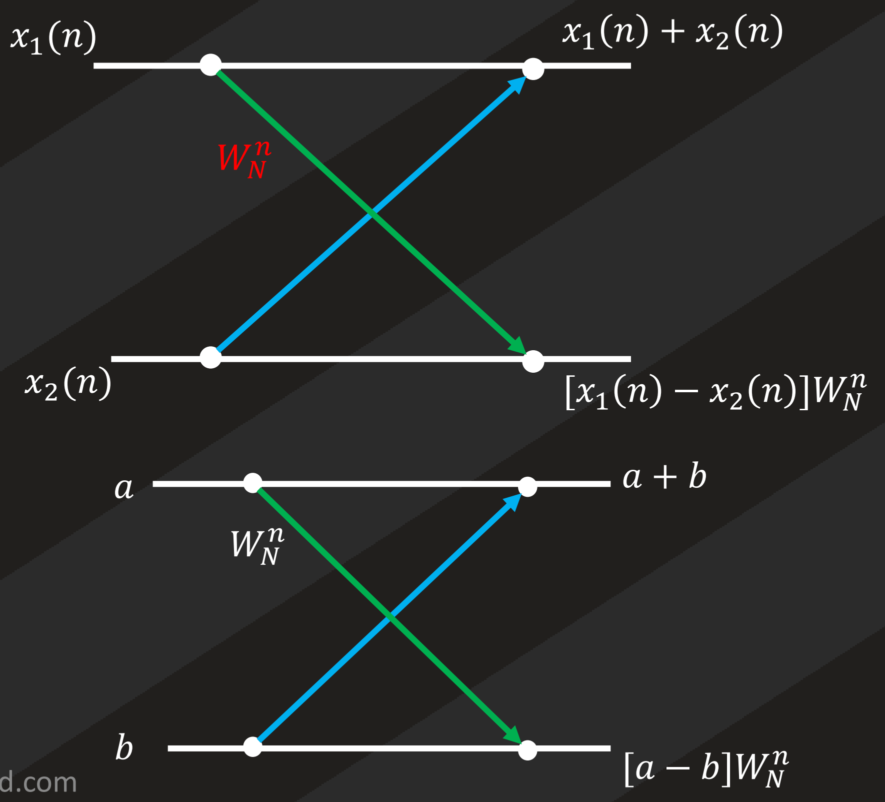
$X(0) = x_1(0) + x_2(0)$ $\quad$ $X(4) = [x_1(0) - x_2(0)]W_8^0$

$X(1) = x_1(1) + x_2(1)$ $\quad$ $X(5) = [x_1(1) - x_2(1)]W_8^1$

$X(2) = x_1(2) + x_2(2)$ $\quad$ $X(6) = [x_1(2) - x_2(2)]W_8^2$

$X(3) = x_1(3) + x_2(3)$ $\quad$ $X(7) = [x_1(3) - x_2(3)]W_8^3$

This operation can be represented by a butterfly diagram



Decimation in Frequency (DIF)

Steps to follow

Step 1 : Find the number of input samples (N)

Step 2 : input sequence in normal order

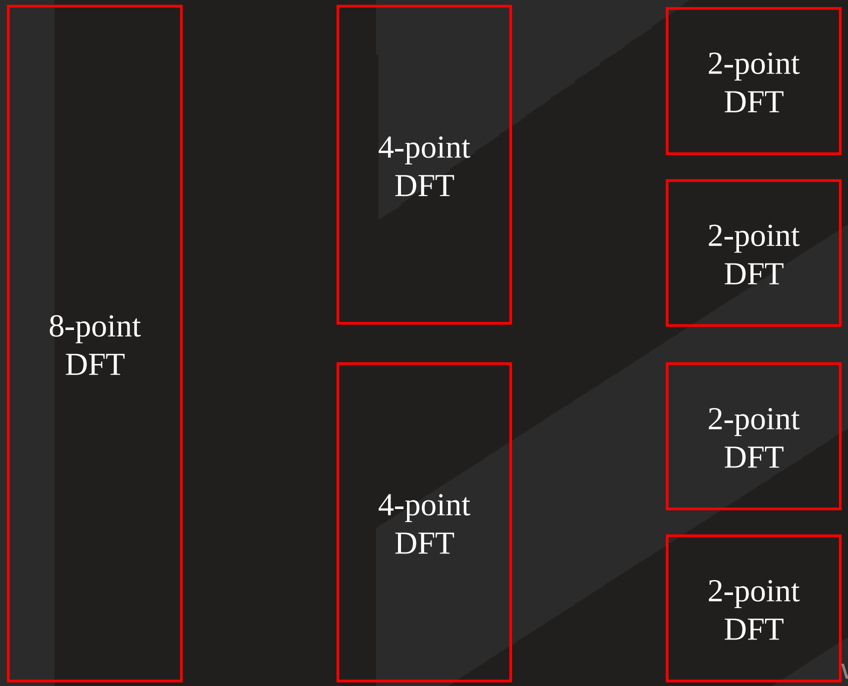
Step 3 : Calculate the number of stages ($M = \log_2 N$)

Step 4 : Calculate the number of max butterflies in stage (N/2)

Step 5 : Calculate the twiddle factor

Step 6 : Evaluate the N point DFT using butterfly diagram

Step7 : The DFT output is in bit-reversed order



Input	Binary	Bit-reversed	Revised samples
x(0)	000	000	x(0)
x(1)	001	100	x(4)
x(2)	010	010	x(2)
x(3)	011	110	x(6)
x(4)	100	001	x(1)
x(5)	101	101	x(5)
x(6)	110	011	x(3)
x(7)	111	111	x(7)

Decimation in Frequency (DIF)

Q) Find the DFT of a sequence x(n) = {0, 1, 2, 3} using DIF algorithm

Solution

Step 1 : Find the number of input samples (N)

N=4

Step 2 : input sequence in normal order

Input
x(0)
x(1)
x(2)
x(3)

Step 3 : Calculate the number of stages $(M = \log_2 N)$

$$M = \log_2 N \quad = \log_2 4$$

$$M = 2$$

Step 4 : Calculate the number of max butterflies in stage

$$\frac{N}{2} = \frac{4}{2} = 2$$

Decimation in Frequency (DIF)

Step 5 : Calculate the twiddle factor

$$k = \frac{Nt}{2^{M-m+1}} \qquad t = 0, 1, 2, \dots 2^{M-m} - 1$$

Stage =1 $(M = 2, m = 1)$

$t = 0,1$

for t=0

$$k = \frac{Nt}{2^{M-m+1}} = \frac{4.0}{2^2} = 0$$

W_4^0 W_4^1

$$W_4^0 = e^{\left(-\frac{j2\pi}{4}\right)0} = 1$$

for t=1

$$k = \frac{4.1}{2^2} = 1$$

$$W_4^1 = e^{\left(-\frac{j2\pi}{4}\right).2} = e^{-j\pi} = \cos(\pi) - j\sin(\pi)$$

$$W_4^1 = -j$$

Stage =2 $(M = 2, m = 2)$

$t = 0$

$$k = \frac{Nt}{2^{M-m+1}} = \frac{4.0}{2^1} = 0$$

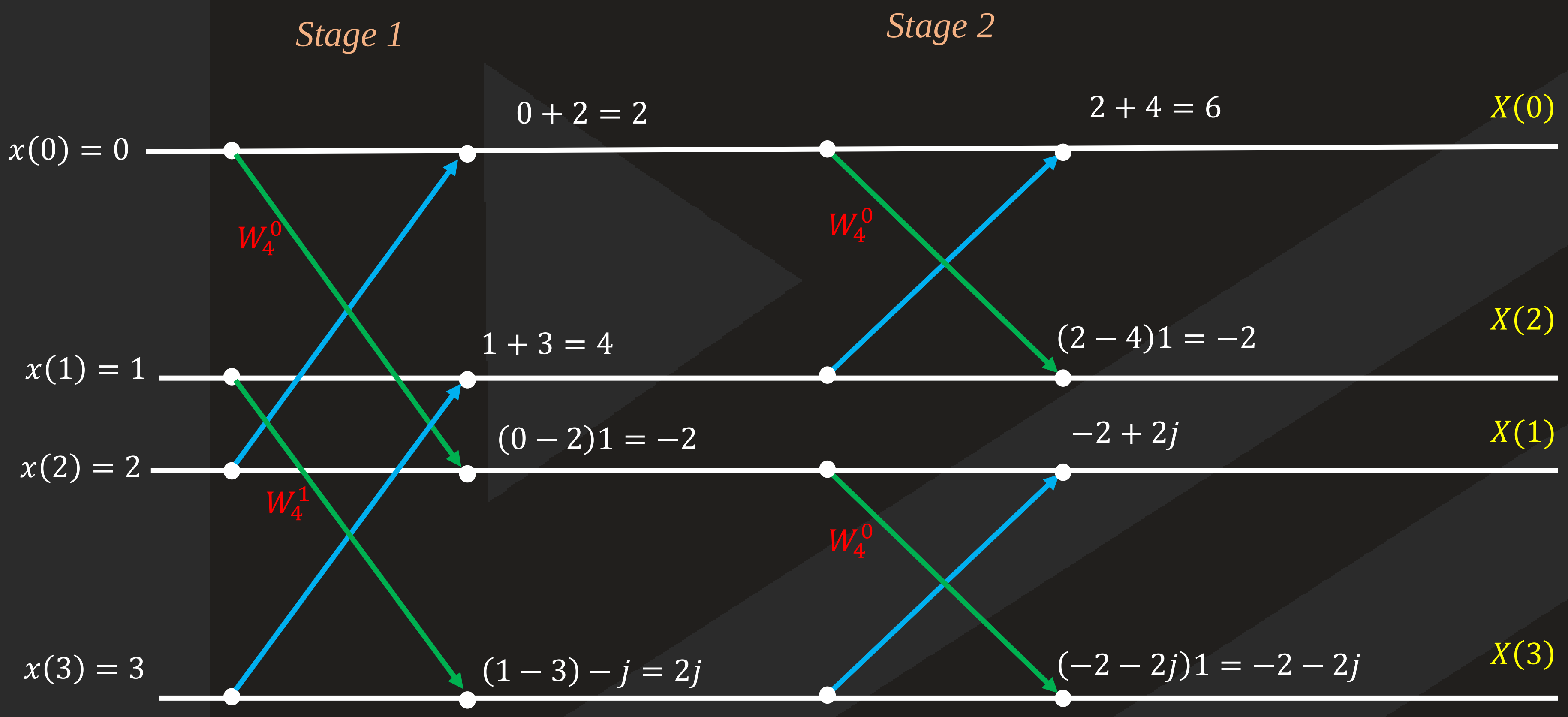
W_4^0

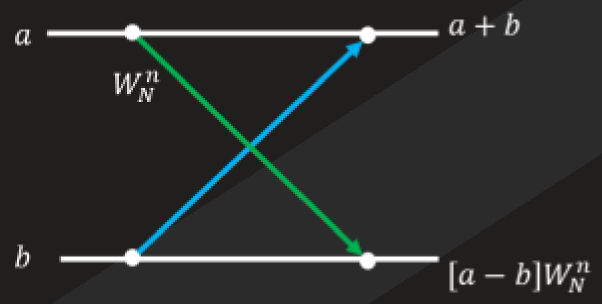
Decimation in Frequency (DIF)

Step 6 : Evaluate the N point DFT using butterfly diagram

$x(n) = \{0, 1, 2, 3\}$

$W_4^0 = 1 \quad W_4^1 = -j$





Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)

Step7 : The DFT output is in bit-reversed order

$X(k) = \{6, -2 + 2j, -2, -2 - 2j\}$

Decimation in Frequency (DIF)

Q) Find the DFT of a sequence x(n) = {1, 2, 3, 4, 4, 3, 2, 1 } using DIF algorithm

Solution

Step 1 : Find the number of input samples (N)

N=8

Step 2 : input sequence in normal order

Input	Binary	Bit-reversed	Revised samples
x(0)	000	000	x(0)
x(1)	001	100	x(4)
x(2)	010	010	x(2)
x(3)	011	110	x(6)
x(4)	100	001	x(1)
x(5)	101	101	x(5)
x(6)	110	011	x(3)
x(7)	111	111	x(7)

Step 3 : Calculate the number of stages $(M = \log_2 N)$

$$M = \log_2 N \quad = \log_2 8$$

$$M = 3$$

Step 4 : Calculate the number of max butterflies in stage

$$\frac{N}{2} = \frac{8}{2} = 4$$

Decimation in Frequency (DIF)

$W_8^3 = e^{\left(\frac{-j2\pi}{8}\right)3} = e^{\frac{-j3\pi}{4}}$

$= \cos\left(\frac{3\pi}{4}\right) - j\sin\left(\frac{3\pi}{4}\right)$

$W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} = 0.7071 - j0.7071$

$W_8^3 = -0.7071 - j0.7071$

Step 5 : Calculate the twiddle factor

$$k = \frac{Nt}{2^{M-m+1}} \qquad t = 0, 1, 2, \dots 2^{M-m} - 1$$

<u>*Stage =1*</u> $(M = 3, m = 1)$

$t = 0,1,2,3$

<u>*for t=0*</u>

$k = \frac{8.0}{2^3} = 0$ $\quad W_8^0 \quad$ $W_8^0 = 1$

<u>*for t=1*</u>

$k = \frac{8.1}{2^3} = 1$ $\quad W_8^1 \quad$ $W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

<u>*for t=2*</u>

$k = \frac{8.2}{2^3} = 2$ $\quad W_8^2 \quad$ $W_8^2 = -j$

<u>*for t=3*</u>

$k = \frac{8.3}{2^3} = 3$ $\quad W_8^3 \quad$ $W_8^2 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

<u>*Stage =2*</u> $(M = 3, m = 2)$

$t = 0,1$

<u>*for t=0*</u>

$k = \frac{8.0}{2^2} = 0$ $\quad W_8^0 \quad$ $W_8^0 = 1$

<u>*for t=1*</u>

$k = \frac{8.1}{2^2} = 2$ $\quad W_8^2 \quad$ $W_8^2 = -j$

<u>*Stage =3*</u> $(M = 3, m = 3)$

$t = 0$

<u>*for t=0*</u>

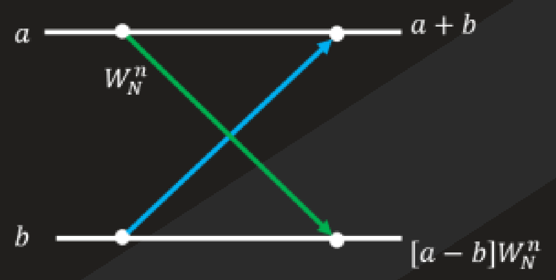
$k = \frac{8.0}{2^1} = 0$ $\quad W_8^0 \quad$ $W_8^0 = 1$

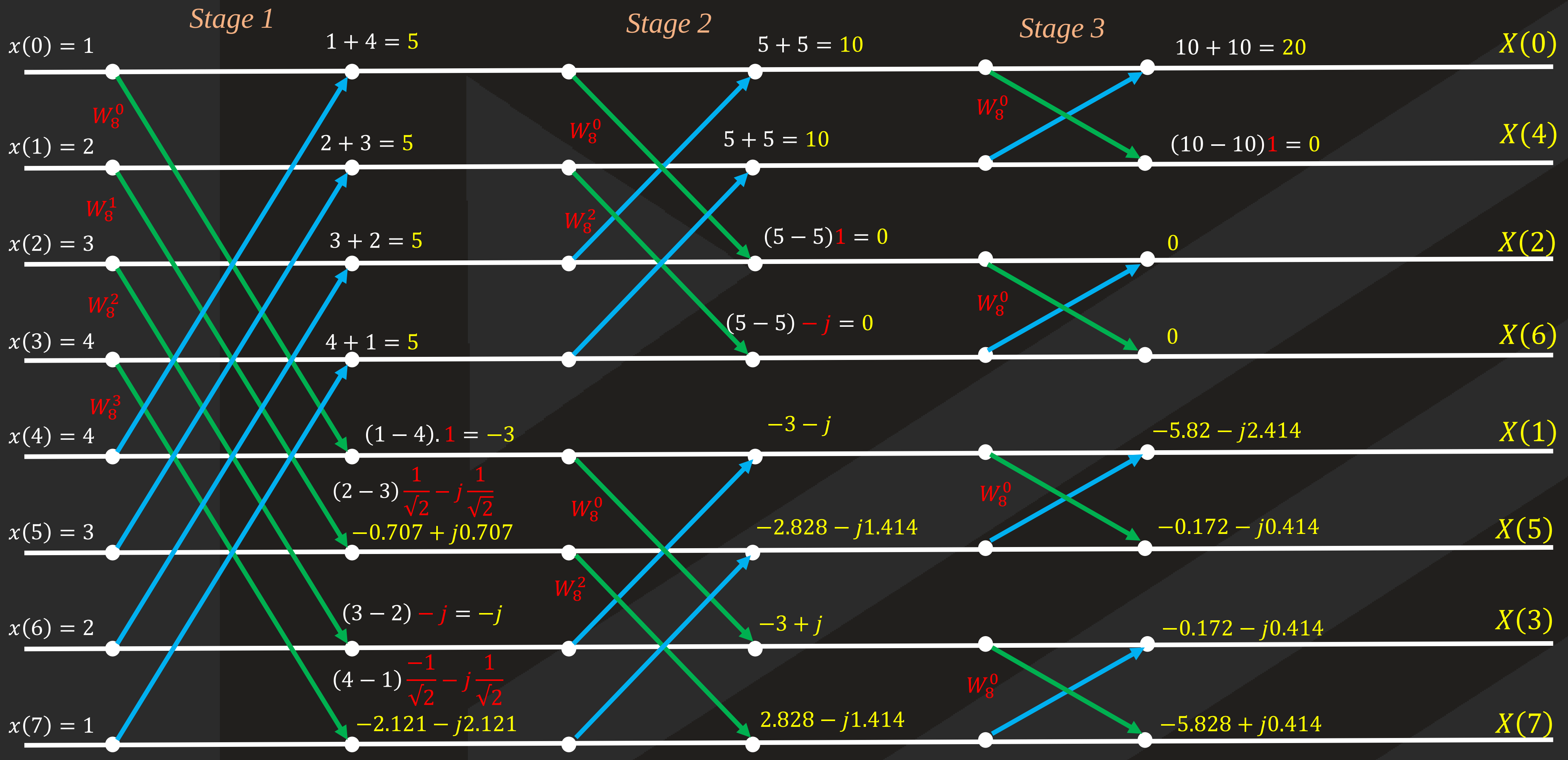
Decimation in Frequency (DIF)

Step 6 : Evaluate the N point DFT using butterfly diagram

$x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$

$W_8^0 = 1$ $W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$ $W_8^2 = -j$ $W_8^3 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$





Input	Revised samples
x(0)	x(0)
x(1)	x(4)
x(2)	x(2)
x(3)	x(6)
x(4)	x(1)
x(5)	x(5)
x(6)	x(3)
x(7)	x(7)

Step7 : The DFT output is in bit reversed order

$X(k) = \{20, -5.82 - j2.414, 0, -0.172 - j0.414, 0, -0.172 - j0.414, 0, -5.828 + j0.414\}$

IDFT Computation using Radix -2 FFT algorithm

The inverse DFT of an N-point sequence X(k), for k=0,1,2,…, N-1

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k)W_N^{nk}\right]^*$$

Q) Find the IDFT of the sequence X(k) = {10, -2+2j, -2, -2-2j} using DIT algorithm

Solution

Step 1 : Find the number of input samples (N)

N=4

Step 2 : Bir reversal

Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)

Step 3 : Calculate the number of stages $(M = \log_2 N)$

$$M = \log_2 N \quad = \log_2 4$$

$$M = 2$$

Step 4 : Calculate the number of max butterflies in stage

$$\frac{N}{2} = \frac{4}{2} = 2$$

IDFT Computation using Radix -2 FFT algorithm

Step 5 : Calculate the twiddle factor

$$k = \frac{Nt}{2^M} \qquad t = 0, 1, 2, \ldots 2^{M-1} - 1$$

Stage =1 $(M = 1)$

$t = 0$

$$k = \frac{Nt}{2^M} = \frac{4.0}{2^1} = 0$$

W_4^0

$$W_4^0 = e^{\left(-\frac{j2\pi}{4}\right)0} = 1$$

Stage =2 $(M = 2)$

$t = 0,1$

for t=0

$$k = \frac{4.0}{2^2} = 0 \qquad W_4^0$$

for t=1

$$k = \frac{4.1}{2^2} = 1 \qquad W_4^1$$

$$W_4^1 = e^{\left(-\frac{j2\pi}{4}\right)2} = e^{-j\pi} = \cos(\pi) - j\sin(\pi)$$

$$W_4^1 = -j$$

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Step 6 : Find the conjugates of X(k)

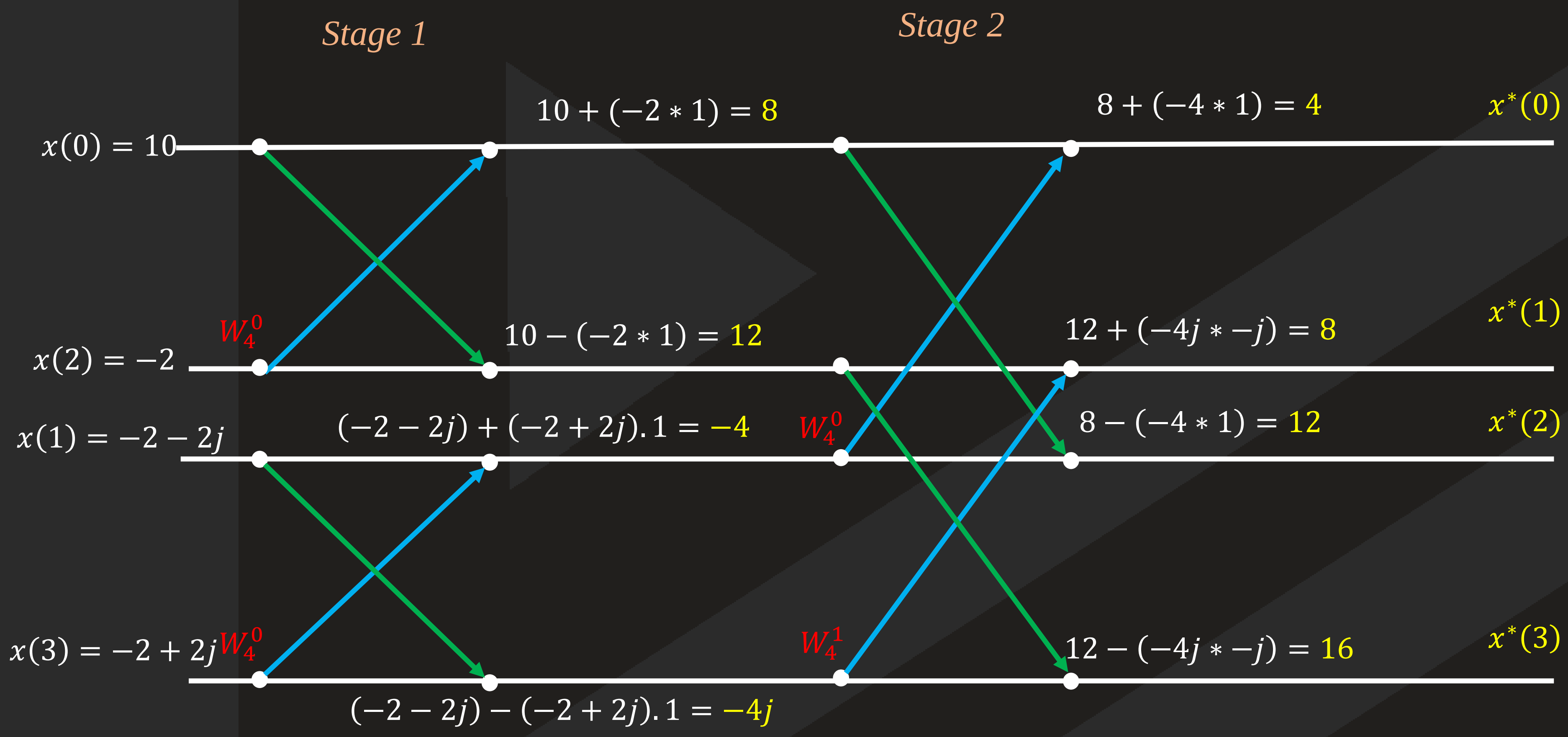
$X(k)$ = {10, -2+2j, -2, -2-2j}

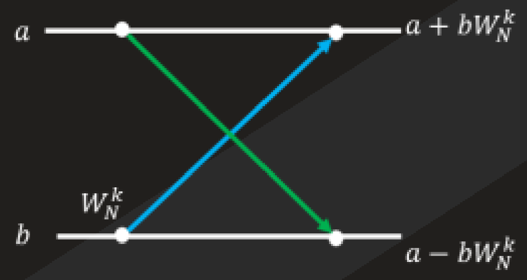
$X^*(k)$ = {10, -2-2j, -2, -2+2j}

IDFT Computation using Radix -2 FFT algorithm

Step 7 : Evaluate the IDFT using butterfly diagram

$X^*(k) = \{10, -2-2j, -2, -2+2j\}$ $\quad W_4^0 = 1 \quad W_4^1 = -j$





Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)

Step8 : The output is in normal order and divide it with N

$$x^*(n) = \frac{1}{4}\{4, 8, 12, 16\} \qquad x(n) = \{1, 2, 3, 4\}$$

IDFT Computation using Radix -2 FFT algorithm

Q) Find the IDFT of the sequence X(k) = {7, 2, 3, 1+j} using DIF algorithm

Solution

Step 1 : Find the number of input samples (N)

N=4

Step 2 : Bit reversal

Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)

Step 3 : Calculate the number of stages ($M = \log_2 N$)

$$M = \log_2 N \quad = \log_2 4$$

$$M = 2$$

Step 4 : Calculate the number of max butterflies in stage

$$\frac{N}{2} = \frac{4}{2} = 2$$

IDFT Computation using Radix -2 FFT algorithm

Step 5 : Calculate the twiddle factor

$k = \frac{Nt}{2^{M-m+1}}$ $t = 0, 1, 2, \ldots 2^{M-m} - 1$

Stage =1 $(M = 2, m = 1)$

$t = 0,1$

for t=0

$k = \frac{4.0}{2^2} = 0$ W_4^0

for t=1

$k = \frac{4.1}{2^2} = 1$ W_4^1

$$W_4^0 = e^{\left(-\frac{j2\pi}{4}\right)0} = 1$$

$W_4^1 = e^{\left(-\frac{j2\pi}{4}\right).2}$ $= e^{-j\pi}$ $= \cos(\pi) - j\sin(\pi)$

$$W_4^1 = -j$$

Stage =2 $(M = 2, m = 2)$

$t = 0$

$k = \frac{Nt}{2^M}$ $= \frac{4.0}{2^1}$ $= 0$

W_4^0

$$W_4^0 = e^{\left(-\frac{j2\pi}{4}\right)0} = 1$$

Step 6 : Find the conjugates of X(k)

X(k) = {7, 2, 3, 1+j}

X(k) = {7, 2, 3, 1-j}*

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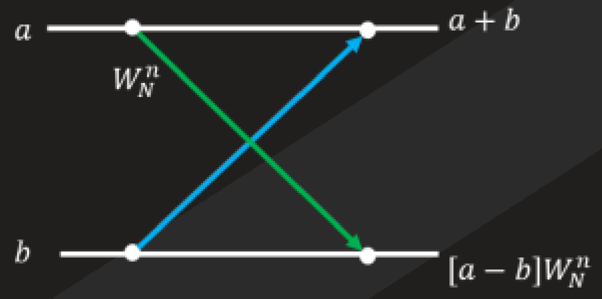
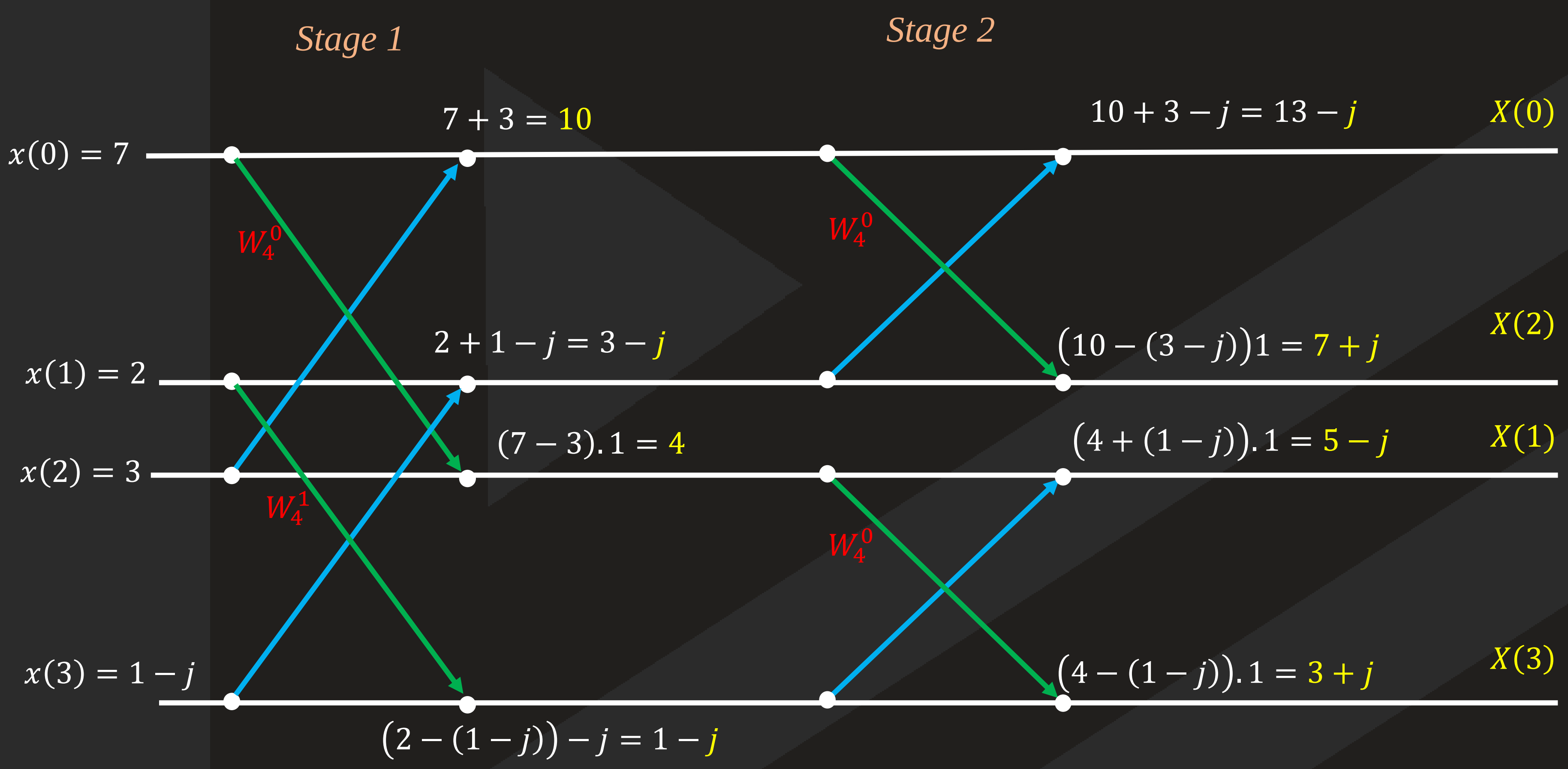
Decimation in Frequency (DIF)

Step 7 : Evaluate the N point DFT using butterfly diagram

$X^*(k) = \{7, 2, 3, 1-j\}$ $W_4^0 = 1$ $W_4^1 = -j$



Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)



Step8 : The output is in normal order and divide it with N

$$x^*(n) = \frac{1}{4}\{13-j, 5-j, 7+j, 3+j\}$$

Application of FFT

Efficient computation of DFT of two real sequences

Let $x_1(n)$ and $x_2(n)$ are two real sequences of length N and let x(n) be a complex values sequence defined as $x(n) = x_1(n)+jx_2(n)$

Now find the DFT of the sequence x(n) which is linear

$$X(k) = X_1(k) + jX_2(k)$$

The sequences $x_1(n)$ and $x_2(n)$ can be expressed in terms of x(n) as

$$x_1(n) = \frac{x(n) + x^*(n)}{2} \qquad x_2(n) = \frac{x(n) - x^*(n)}{2j}$$

Then the DFT of $x_1(n)$ and $x_2(n)$ are

$$X_1(k) = \frac{1}{2}\{DFT[x(n)] + DFT[x^*(n)]\} \qquad X_2(k) = \frac{1}{2j}\{DFT[x(n)] - DFT[x^*(n)]\}$$

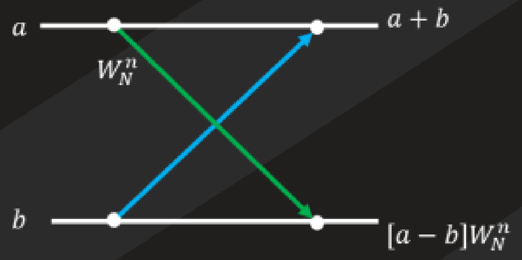
From conjugation property of twiddle factor

$$x^*(n) \overset{DFT}{\Longleftrightarrow} X^*(N-k)$$

$$X_1(k) = \frac{1}{2}\{X(k) + X^*(N-k)\} \qquad X_2(k) = \frac{1}{2j}\{X(k) - X^*(N-k)\}$$

Efficient computation of DFT of two real sequences

Q) Find the DFT of two sequence $x_1(n) = \{1,3,1,2\}$ and $x_2(n) = \{2,5,1,3\}$



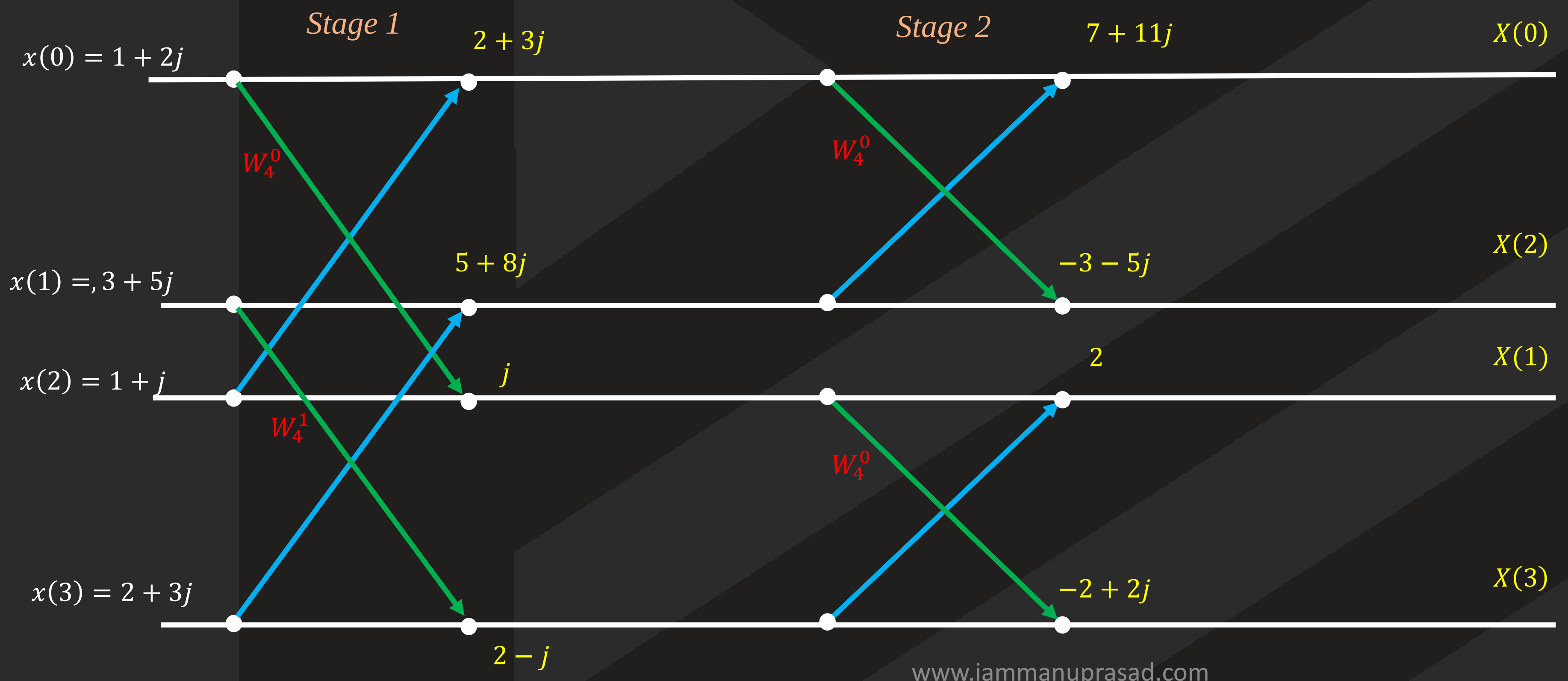
Solution

$$x(n) = x_1(n) + jx_2(n) \quad \text{For n = 0,1,2,3} \qquad x(n) = \{1+2j, 3+5j, 1+j, 2+3j\}$$

Now find the DFT of the sequence x(n) using DIT or DIF method

$$x(n) = \{1+2j, 3+5j, 1+j, 2+3j\} \qquad W_4^0 = 1 \qquad W_4^1 = -j$$

Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)



$$X(k) = \{7+11j, 2, -3-5j, -2+2j\}$$

$$X(k) = \{7 + 11j, 2, -3 - 5j, -2 + 2j\}$$

Now we have to calculate $X_1(k)$ and $X_2(k)$

$$X_1(k) = \frac{1}{2}\{X(k) + X^*(N-k)\}$$
$$X_2(k) = \frac{1}{2j}\{X(k) - X^*(N-k)\}$$

For k=0

$$X_1(0) = \frac{1}{2}\{X(0) + X^*(4-0)\}$$
$$= \frac{1}{2}\{7 + 11j + 7 - 11j\} \quad = 7$$

For k=1

$$X_1(1) = \frac{1}{2}\{X(1) + X^*(4-1)\}$$
$$= \frac{1}{2}\{2 + (-2 - 2j)\} \quad = -j$$

For k=2

$$X_1(2) = \frac{1}{2}\{X(2) + X^*(4-2)\}$$
$$= \frac{1}{2}\{-3 - 5j + (-3 + 5j)\} \quad = -3$$

For k=3

$$X_1(3) = \frac{1}{2}\{X(3) + X^*(4-3)\}$$
$$= \frac{1}{2}\{-2 + 2j + (2)\} \quad = j$$

For k=0

$$X_2(0) = \frac{1}{2j}\{X(0) - X^*(4-0)\}$$
$$= \frac{1}{2j}\{7 + 11j - 7 - 11j\} \quad = 11$$

For k=1

$$X_2(1) = \frac{1}{2j}\{X(1) - X^*(4-1)\}$$
$$= \frac{1}{2j}\{2 - (-2 - 2j)\} \quad = -2j + 1$$

For k=2

$$X_2(2) = \frac{1}{2j}\{X(2) - X^*(4-2)\}$$
$$= \frac{1}{2j}\{-3 - 5j - (-3 + 5j)\} \quad = -5$$

For k=3

$$X_2(3) = \frac{1}{2j}\{X(3) - X^*(4-3)\}$$
$$= \frac{1}{2}\{-2 + 2j - (2)\} \quad = 2j + 1$$

$$X_1(k) = \{7, -j, -3, j\}$$

$$X_2(k) = \{11, 1 - 2j, -5, 1 + 2j\}$$

Application of FFT

Efficient computation of DFT of a 2N-point real sequence

Let g(n) is a real valued sequences of 2N points.

To find the 2N point DFT from N point DFT , we divide the sequence to two

$$x_1(n) = g(2n) \qquad x_2(n) = g(2n+1)$$

Now follow same as the DFT computation of two real sequence

$$X_1(k) = \frac{1}{2}\{X(k) + X^*(N-k)\} \qquad X_2(k) = \frac{1}{2j}\{X(k) - X^*(N-k)\}$$

Finally we must express the 2N point DT in terms of two N point DFTs

$$G(k) = \sum_{n=0}^{N-1} g(2n)W_{2N}^{2nk} + \sum_{n=0}^{N-1} g(2n+1)W_{2N}^{(2n+1)k}$$

$$= \sum_{n=0}^{N-1} x_1(n)W_{2N}^{2nk} + W_{2N}^{k} \sum_{n=0}^{N-1} x_2(n)W_{2N}^{2nk}$$

$$W_N^2 = e^{\left(-\frac{j2\pi}{N}\right)2} = e^{\left(-\frac{j2\pi}{\frac{N}{2}}\right)} = W_{\frac{N}{2}}$$

$$W_{2N}^2 = e^{\left(-\frac{j2\pi}{2N}\right)2} = W_N$$

$$G(k) = \sum_{n=0}^{N-1} x_1(n)W_N^{nk} + W_{2N}^{k} \sum_{n=0}^{N-1} x_2(n)W_N^{nk}$$

$$G(k) = X_1(k) + W_{2N}^{k} X_2(k)$$

$$G(k+N) = X_1(k) - W_{2N}^{k} X_2(k)$$

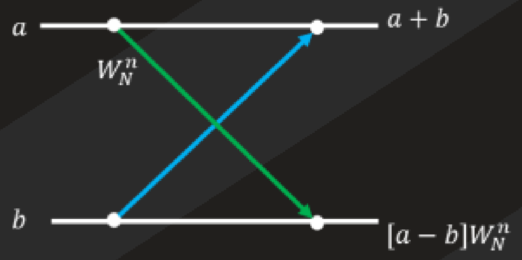
$$Where\ k = 0,1,2,\dots,N-1$$

Efficient computation of DFT of a 2N-point real sequence

Q) Find the DFT of the sequence x(n) = {1,3,7,2,1,2,1,3} using 4-point DFT

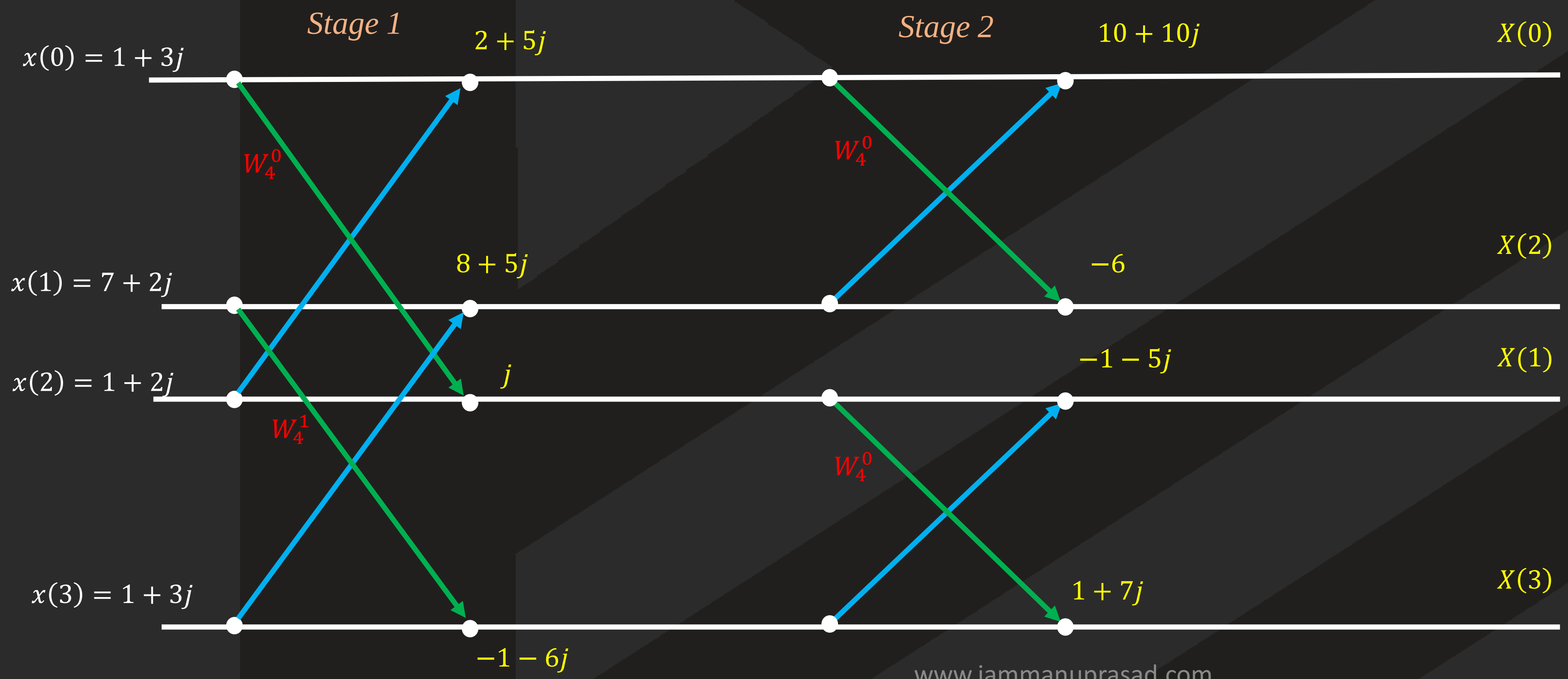
<u>Solution</u> $x_1(n) = \{1,7,1,1,\}$ $x_2(n) = \{3,2,2,3\}$

$x(n) = x_1(n) + jx_2(n)$ $x(n) = \{1+3j, 7+2j, 1+2j, 1+3j\}$



Now find the DFT of the sequence x(n) using DIT or DIF method

$x(n) = \{1+3j, 7+2j, 1+2j, 1+3j\}$ $W_4^0 = 1$ $W_4^1 = -j$



Input	Binary	Bit-reversed	Revised samples
x(0)	00	00	x(0)
x(1)	01	10	x(2)
x(2)	10	01	x(1)
x(3)	11	11	x(3)

$X(k) = \{10+10j, -1-5j, -6, 1+7j\}$

$X(k) = \{10+10j, -1-5j, -6, 1+7j\}$

Now we have to calculate $X_1(k)$ and $X_2(k)$

$$X_1(k) = \frac{1}{2}\{X(k) + X^*(N-k)\}$$
$$X_2(k) = \frac{1}{2j}\{X(k) - X^*(N-k)\}$$

For k=0

$$X_1(0) = \frac{1}{2}\{X(0) + X^*(4-0)\}$$
$$= \frac{1}{2}\{10+10j+10-10j\} \quad = 10$$

For k=1

$$X_1(1) = \frac{1}{2}\{X(1) + X^*(4-1)\}$$
$$= \frac{1}{2}\{-1-5j+1-7j\} = -6j$$

For k=2

$$X_1(2) = \frac{1}{2}\{X(2) + X^*(4-2)\}$$
$$= \frac{1}{2}\{-6+(-6)\} \quad = -6$$

For k=3

$$X_1(3) = \frac{1}{2}\{X(3) + X^*(4-3)\}$$
$$= \frac{1}{2}\{1+7j+(-1+5j)\} \quad = 6j$$

For k=0

$$X_2(0) = \frac{1}{2j}\{X(0) - X^*(4-0)\}$$
$$= \frac{1}{2j}\{10+10j-(10-10j)\} = 10$$

For k=1

$$X_2(1) = \frac{1}{2j}\{X(1) + X^*(4-1)\}$$
$$= \frac{1}{2j}\{-1-5j-(1-7j)\} \quad = 1+j$$

For k=2

$$X_2(2) = \frac{1}{2j}\{X(2) + X^*(4-2)\}$$
$$= \frac{1}{2j}\{-6-(-6)\} \quad = 0$$

For k=3

$$X_2(3) = \frac{1}{2j}\{X(3) - X^*(4-3)\}$$
$$= \frac{1}{2}\{1+7j-(-1+5j)\} \quad = 1-j$$

$$X_1(k) = \{10, -6j, -6, 6j\}$$
$$X_2(k) = \{10, 1+j, 0, 1-j\}$$

$$X_1(k) = \{10, -6j, -6, 6j\}$$

$$X_2(k) = \{10, 1+j, 0, 1-j\}$$

Here 2N =8 so,

$W_8^0 = 1$ $\quad W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$ $\quad W_8^2 = -j$ $\quad W_8^3 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

For k=0

$$X(0) = X_1(0) + W_8^0 X_2(0)$$

$$= 10 + 1.[10] \quad = 20$$

For k=1

$$X(1) = X_1(1) + W_8^1 X_2(1)$$

$$= -6j + [1+j]\left[\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right] \quad = \sqrt{2} - 6j$$

For k=2

$$X(2) = X_1(2) + W_8^2 X_2(2)$$

$$= -6 + 0[-j] \quad = -6$$

For k=3

$$X(3) = X_1(3) + W_8^3 X_2(3)$$

$$= 6j + [1-j]\left[\frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right] \quad = -\sqrt{2} + 6j$$

For k=0

$$X(0+4) = X_1(0) - W_8^0 X_2(0)$$

$$= 10 - 1.[10] \quad = 0$$

For k=1

$$X(1+4) = X_1(1) - W_8^1 X_2(1)$$

$$= -6j - [1+j]\left[\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right] \quad = -\sqrt{2} - 6j$$

For k=2

$$X(2+4) = X_1(2) - W_8^2 X_2(2)$$

$$= -6 - 0[-j] \quad = -6$$

For k=3

$$X(3+4) = X_1(3) - W_8^3 X_2(3)$$

$$= 6j - [1-j]\left[\frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right] \quad = \sqrt{2} + 6j$$

$$G(k) = X_1(k) + W_{2N}^k X_2(k)$$

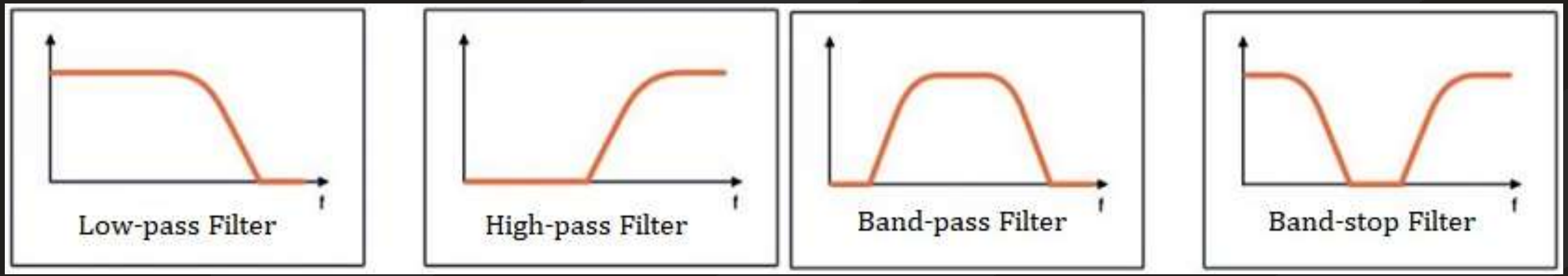
$$G(k+N) = X_1(k) - W_{2N}^k X_2(k)$$

$$X(k) = \{20, \sqrt{2} - 6j, -6, -\sqrt{2} + 6j, 0, -\sqrt{2} - 6j, -6, \sqrt{2} + 6j\}$$

MODULE 3 Finite Impulse Response (FIR) Filters

- What is a Filter?
  - Any medium through which the signal passes, whatever its form, can be regarded as a filter.
  - However, we do not usually think of something as a filter unless it can modify the signal in some way. For example, speaker wire is not considered a filter, but the speaker is
- A digital filter is just a filter that operates on digital signals, such as sound represented inside a computer.
- It is a computation which takes one sequence of numbers (the input signal) and produces a new sequence of numbers (the filtered output signal).

Types of Filter

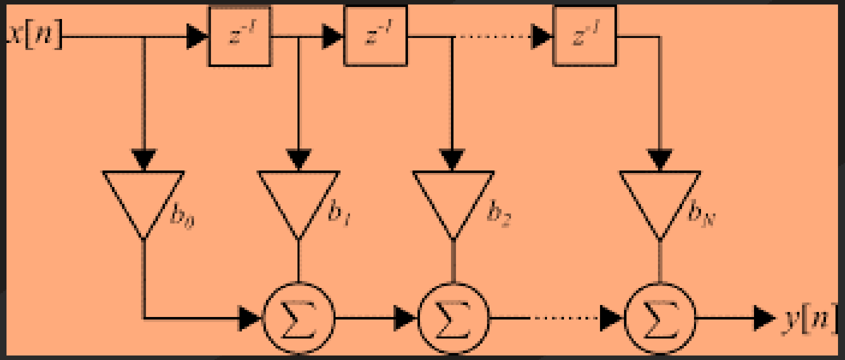


Finite Impulse Response (FIR) Filters

- It is one of two main types of digital filters used in DSP applications.
- FIR filter gets its name because the same number (finite) input values you get going into the filter, you get coming out of the output
- The design methods of FIR filter based on approximation of ideal filter
- Properties of FIR filter
    - *Require no feedback:* This means that any rounding errors are not compounded by summed iterations. The same relative error occurs in each calculation. This also makes implementation simpler.
    - *Inherent stability:* This is due to the fact that, because there is no required feedback, all the poles are located at the origin and thus are located within the unit circle (the required condition for stability in a Z transformed system).
    - *Phase Issue:* can easily be designed to be linear phase by making the coefficient sequence symmetric; linear phase, or phase change proportional to frequency, corresponds to equal delay at all frequencies. This property is sometimes desired for phase-sensitive applications, for example data communications, crossover filters, and mastering.
- The main disadvantage of FIR filters is that considerably more computation power

Finite Impulse Response (FIR) Filters

- A discrete-time FIR filter of order *N*. The top part is an N-stage delay line with *N + 1* taps. Each unit delay is a z^{-1} operator in the Z-transform notation.
- The output *y* of a linear time invariant system is determined by convolving its input signal *x* with its impulse response *b*.
- For a discrete-time FIR filter, the output is a weighted sum of the current and a finite number of previous values of the input.
- The operation is described by the following equation, which defines the output sequence *y[n]* in terms of its input sequence *x[n]*:



$$y(n) = b_0x[n] + b_1x[n-1] + b_2x[n-2] + \ ... + b_Nx[n-N]$$

$$y(n) = \sum_{k=0}^{N-1} b_k x(n-k)$$

- *x(n)* : is the input sequence
- y(n) : is the output sequence
- b_k : filter coefficients that make up the impulse response
- N: is the filter order

FIR Impulse response

$$y(n) = \sum_{k=0}^{N-1} b_k x(n-k) \qquad h(n) = \sum_{k=0}^{N-1} b_k \delta[n-k]$$

The Z-transform of the impulse response yields the transfer function of the FIR filter

$$\mathrm{H}(z) = Z\{h(n)\}$$

$$= \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$H(z) = \sum_{n=0}^{N-1} b_n z^{-n}$$

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Linear phase FIR filter – Symmetric Impulse response

Let h(n) be an impulse response of a system then its Fourier transform can be expressed as

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n} \quad \text{------------------ (1)}$$

Since $H(e^{j\omega n})$ is a complex value for linear phase FIR filter , then we can represent it in terms of magnitude and phase

$$H(e^{j\omega}) = \pm|H(e^{j\omega})|e^{-j\alpha\omega} \quad \text{------------------ (2)}$$

Equating (1) & (2)

$$\sum_{n=0}^{N-1} h(n)e^{-j\omega n} = \pm|H(e^{j\omega})|e^{-j\alpha\omega}$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\sum_{n=0}^{N-1} h(n)[\cos\omega n - j\sin\omega n] = \pm|H(e^{j\omega})|[\cos\alpha\omega - j\sin\alpha\omega]$$

Equating sin and cos terms

$$\sum_{n=0}^{N-1} h(n)[\cos\omega n] = \pm|H(e^{j\omega})|[\cos\alpha\omega] \quad \text{---------------- (3)}$$

$$\sum_{n=0}^{N-1} h(n)[\sin\omega n] = \pm|H(e^{j\omega})|[\sin\alpha\omega] \quad \text{---------------- (4)}$$

(4) / (3)

$$\frac{\sum_{n=0}^{N-1} h(n)[\sin\omega n]}{\sum_{n=0}^{N-1} h(n)[\cos\omega n]} = \frac{\sin\alpha\omega}{\cos\alpha\omega}$$

$$\sum_{n=0}^{N-1} h(n)\sin\omega n\cos\alpha\omega = \sum_{n=0}^{N-1} h(n)\cos\omega n\sin\alpha\omega$$

$$0 = \sum_{n=0}^{N-1} h(n)[\cos\omega n\sin\alpha\omega - \sin\omega n\cos\alpha\omega]$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\sum_{n=0}^{N-1} h(n)[\sin(\alpha - n)\omega] = 0$$

The above equation will be zero when

$$h(n) = h(N-1-n)$$

$$\alpha = \frac{N-1}{2}$$

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Linear phase FIR filter – Symmetric Impulse response

The expression for Phase delay and group delay are

$$\tau_p = \frac{-\theta(\omega)}{\omega} \qquad \tau_g = \frac{-d\theta(\omega)}{d\omega}$$

For FIR filter

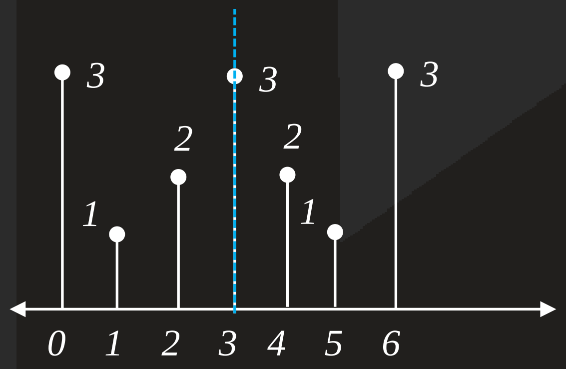
$$\theta(\omega) = -\alpha\omega \ , \qquad -\pi \le \omega \le \pi$$

$$h(n) = h(N-1-n)$$

$$\alpha = \frac{N-1}{2}$$

From the equations and the conditions we can conclude that FIR filter will have constant phase and group delays when the impulse response is symmetrical about $\alpha = \frac{N-1}{2}$

For N is odd



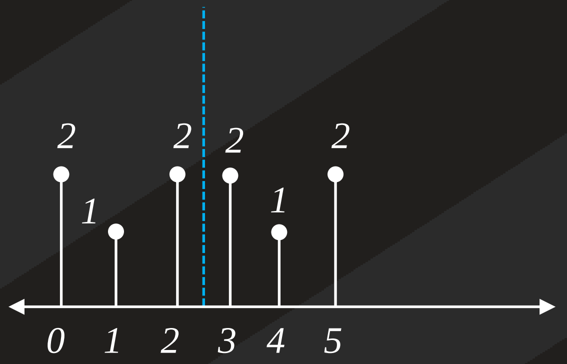
$$\alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3$$

$$h(n) = h(N-1-n)$$

$$h(5) = h(7-1-5)$$

$$= h(1)$$

For N is even



$$\alpha = \frac{N-1}{2} = \frac{6-1}{2} = 2.5$$

$$h(n) = h(N-1-n)$$

$$h(5) = h(6-1-5)$$

$$= h(0)$$

Linear phase FIR filter – Antisymmetric Impulse response

$\theta(\omega) = \beta - \alpha\omega$

Let h(n) be an impulse response of a system then its Fourier transform can be expressed as

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n} \quad \text{------------------ (1)}$$

Since $H(e^{j\omega n})$ is a complex value for linear phase FIR filter , then we can represent it in terms of magnitude and phase. If only constant group delay is required

$$H(e^{j\omega}) = \pm|H(e^{j\omega})|e^{-j(\beta-\alpha\omega)} \quad \text{------------------ (2)}$$

Equating (1) & (2)

$$\sum_{n=0}^{N-1} h(n)e^{-j\omega n} = \pm|H(e^{j\omega})|e^{-j(\beta-\alpha\omega)}$$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$$\sum_{n=0}^{N-1} h(n)[\cos\omega n - j\sin\omega n] = \pm|H(e^{j\omega})|[\cos(\beta-\alpha\omega) - j\sin(\beta-\alpha\omega)]$$

Equating sin and cos terms

$$\sum_{n=0}^{N-1} h(n)[\cos\omega n] = \pm|H(e^{j\omega})|[\cos(\beta-\alpha\omega)] \quad \text{----------------- (3)}$$

$$\sum_{n=0}^{N-1} h(n)[\sin\omega n] = \pm|H(e^{j\omega})|[\sin(\beta-\alpha\omega)] \quad \text{----------------- (4)}$$

(4) / (3)

$$\frac{\sum_{n=0}^{N-1} h(n)[\sin\omega n]}{\sum_{n=0}^{N-1} h(n)[\cos\omega n]} = \frac{\sin(\beta-\alpha\omega)}{\cos(\beta-\alpha\omega)}$$

$$\sum_{n=0}^{N-1} h(n)\sin\omega n\cos(\beta-\alpha\omega) = \sum_{n=0}^{N-1} h(n)\cos\omega n\sin(\beta-\alpha\omega)$$

$$0 = \sum_{n=0}^{N-1} h(n)[\cos\omega n\sin(\beta-\alpha\omega) - \sin\omega n\cos(\beta-\alpha\omega)]$$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$

$$\sum_{n=0}^{N-1} h(n)[\sin(\beta-(\alpha-n)\omega)] = 0$$

$$\beta = \frac{\pi}{2}$$

The equation will be zero when

$$\sum_{n=0}^{N-1} h(n)[\cos(\alpha-n)\omega] = 0$$

$$h(n) = -h(N-1-n)$$

$$\alpha = \frac{N-1}{2}$$

Linear phase FIR filter – Asymmetric Impulse response

The expression for Phase delay and group delay are

$$\tau_p = \frac{-\theta(\omega)}{\omega} \qquad \tau_g = \frac{-d\theta(\omega)}{d\omega}$$

For FIR filter

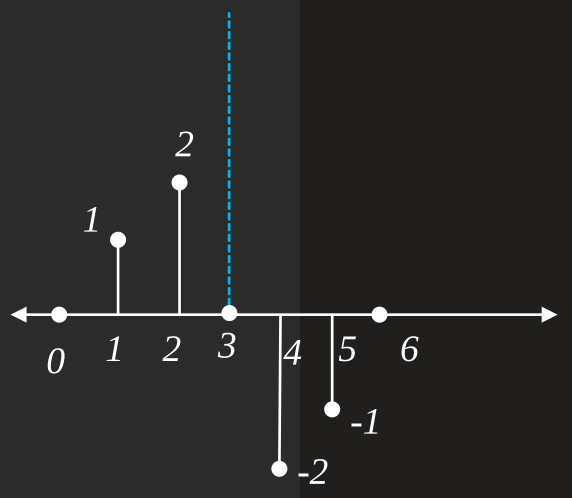
$$\theta(\omega) = \beta - \alpha\omega \;, \qquad -\pi \le \omega \le \pi$$

$$h(n) = -h(N-1-n)$$

$$\alpha = \frac{N-1}{2}$$

From the equations and the conditions we can conclude that FIR filter will have constant group delay and not constant phase delay

For N is odd



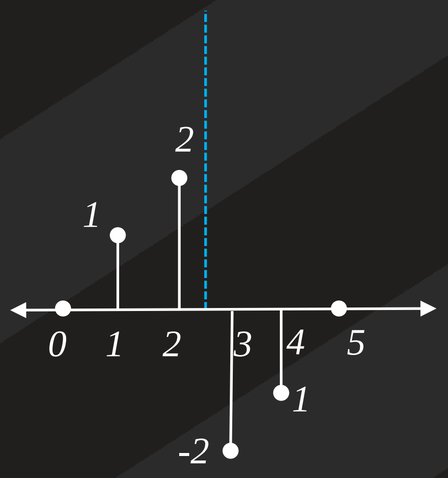
$$\alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3$$

$$h(n) = -h(N-1-n)$$

$$h(5) = -h(7-1-5)$$

$$= -h(1)$$

For N is even



$$\alpha = \frac{N-1}{2} = \frac{6-1}{2} = 2.5$$

$$h(n) = -h(N-1-n)$$

$$h(5) = -h(6-1-5)$$

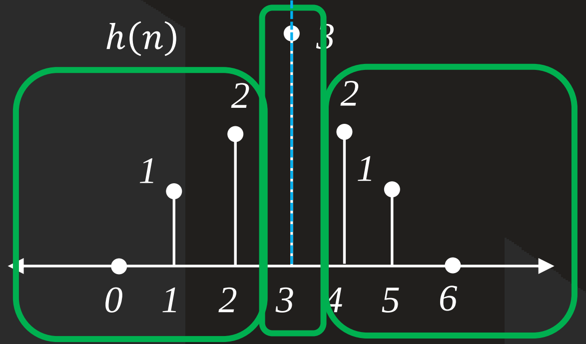
$$= h(0)$$

Frequency response of Linear phase FIR filters

Depending on the value of N and the type of symmetry of filter impulse response sequence there are mainly 4 types of linear phase FIR filter

- Symmetrical impulse response , N=odd
- Symmetrical impulse response , N=even
- Antisymmetric impulse response , N=odd
- Antisymmetric impulse response , N=even

Case1 : Symmetrical impulse response and N - odd



$N = 7$

$\alpha = \frac{N-1}{2} = 3$

Given h(n) and find the Fourier transform $H(e^{j\omega})$

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n}$$

Since N is odd the centre of symmetry will be at n $= \frac{N-1}{2}$

Now lets split the equation into three parts

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)e^{-j\omega n}$$

---------------- (1)

To arrange the limit we assume

$m = N - 1 - n$

$n = N - 1 - m$

When $n = \frac{N+1}{2}$

$\frac{N+1}{2} = N - 1 - m$

$m = \frac{N-3}{2}$

When $n = N - 1$

$N - 1 = N - 1 - m$

$m = 0$

Substitute in (1)

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} h(N-1-m)e^{-j\omega(N-1-m)}$$

Put m=n

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n)e^{-j\omega(N-1-n)}$$

For symmetric impulse response $h(n) = h(N-1-n)$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega(N-1-n)}$$

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Case1 : Symmetrical impulse response and N - odd

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega(N-1-n)}$$

Taking $e^{-j\omega\left(\frac{N-1}{2}\right)}$ outside

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[e^{-j\omega n}.e^{j\omega\left(\frac{N-1}{2}\right)} + e^{-j\omega(N-1-n)}.e^{j\omega\left(\frac{N-1}{2}\right)}\right]\right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(N-1-n-\left(\frac{N-1}{2}\right)\right)}\right]\right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]\right\}$$

$$2\cos\theta = e^{j\theta} + e^{-j\theta}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[2\cos\omega\left(\frac{N-1}{2}-n\right)\right]\right\}$$

Let

$$k = \frac{N-1}{2} - n$$

$$n = \frac{N-1}{2} - k$$

When $n = 0$

$$0 = \frac{N-1}{2} - k$$

$$k = \frac{N-1}{2}$$

When $n = \frac{N-3}{2}$

$$\frac{N-3}{2} = \frac{N-1}{2} - k$$

$$k = 1$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{k=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - k\right)\cos\omega k\right\}$$

Put $k = n$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n\right\}$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=1}^{\frac{N-1}{2}} a(n)\cos\omega n\right\}$$

Where

$$a(0) = h\left(\frac{N-1}{2}\right) \qquad a(n) = 2h\left(\frac{N-1}{2} - n\right)$$

Case1 : Symmetrical impulse response and N - odd

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=1}^{\frac{N-1}{2}} a(n)\cos\omega n\right\}$$

Where

$$a(0) = h\left(\frac{N-1}{2}\right) \quad a(n) = 2h\left(\frac{N-1}{2} - n\right)$$

From this we can express the amplitude and phase function

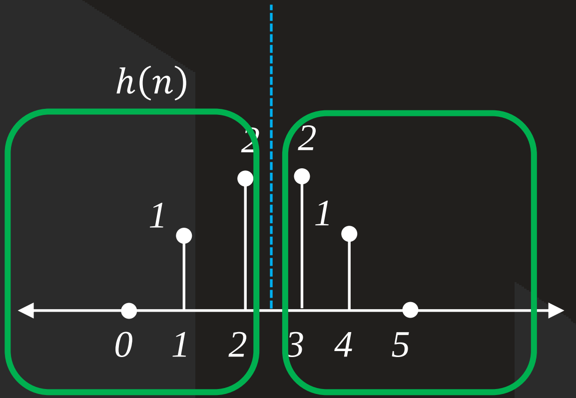
Amplitude

$$|H(e^{j\omega})| = \sum_{n=1}^{\frac{N-1}{2}} a(n)\cos\omega n$$

Phase

$$\angle H(e^{j\omega}) = -\omega\left(\frac{N-1}{2}\right) = -\alpha\omega$$

Case2 : Symmetrical impulse response and N - even



$N = 6$

$$\alpha = \frac{N-1}{2} = 2.5$$

Given h(n) and find the Fourier transform $H(e^{j\omega})$

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n}$$

Since N is even the centre of symmetry will be at $n = \frac{N-1}{2}$

For symmetric impulse response with even number of samples and centre of symmetry lies between $n = \frac{N-2}{2}$ *and* $\frac{N}{2}$

Then we can split the equation into two parts

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n)e^{-j\omega n} \quad \text{--------------- (1)}$$

To arrange the limit we assume

$m = N - 1 - n$

$n = N - 1 - m$

When $n = \frac{N}{2}$

$\frac{N}{2} = N - 1 - m$

$m = \frac{N}{2} - 1$

When $n = N - 1$

$N - 1 = N - 1 - m$

$m = 0$

Substitute in (1)

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} + \sum_{m=0}^{\frac{N-2}{2}} h(N-1-m)e^{-j\omega(N-1-m)}$$

Put m=n

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right)e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=0}^{\frac{N-2}{2}} h(N-1-n)e^{-j\omega(N-1-n)}$$

For symmetric impulse response $h(n) = h(N-1-m)$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N-1}{2}} h(n)e^{-j\omega(N-1-n)}$$

Case2 : Symmetrical impulse response and N - even

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega(N-1-n)}$$

Taking $e^{-j\omega\left(\frac{N-1}{2}\right)}$ outside

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=0}^{\frac{N-2}{2}} h(n)\left[e^{-j\omega n}.e^{j\omega\left(\frac{N-1}{2}\right)} + e^{-j\omega(N-1-n)}.e^{j\omega\left(\frac{N-1}{2}\right)}\right]\right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=0}^{\frac{N-2}{2}} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(N-1-n-\left(\frac{N-1}{2}\right)\right)}\right]\right\}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=0}^{\frac{N-2}{2}} h(n)\left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right]\right\}$$

$2\cos\theta = e^{j\theta} + e^{-j\theta}$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=0}^{\frac{N-2}{2}} h(n)\left[2\cos\omega\left(\frac{N-1}{2}-n\right)\right]\right\}$$

$$\frac{N-1}{2} - n = \frac{N}{2} - n - \frac{1}{2}$$

Let

$$k = \frac{N}{2} - n$$

$$n = \frac{N}{2} - k$$

When $n = 0$

$$0 = \frac{N}{2} - k$$

$$k = \frac{N}{2}$$

When $n = \frac{N-2}{2}$

$$\frac{N-2}{2} = \frac{N}{2} - k$$

$$k = 1$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{k=1}^{\frac{N}{2}} 2h\left(\frac{N}{2}-k\right)\cos\omega\left(k-\frac{1}{2}\right)\right\}$$

Put $k = n$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2}-n\right)\cos\omega\left(n-\frac{1}{2}\right)\right\}$$

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=1}^{\frac{N}{2}} b(n)\cos\omega\left(n-\frac{1}{2}\right)\right\}$$

Where

$$b(n) = 2h\left(\frac{N}{2}-n\right)$$

Case2 : Symmetrical impulse response and N - even

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=1}^{\frac{N}{2}} b(n)\cos\omega\left(n-\frac{1}{2}\right)\right\}$$

Where

$$b(n) = 2h\left(\frac{N}{2}-n\right)$$

From this we can express the amplitude and phase function

Amplitude

$$|H(e^{j\omega})| = \sum_{n=1}^{\frac{N}{2}} b(n)\cos\omega\left(n-\frac{1}{2}\right)$$

Phase

$$\angle H(e^{j\omega}) = -\omega\left(\frac{N-1}{2}\right) = -\alpha\omega$$

Case 3 : Antisymmetrical impulse response and N - odd

In the similar way of symmetric, we get for antisymmetric as

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} e^{\frac{j\pi}{2}} \left\{ \sum_{n=1}^{\frac{N-1}{2}} c(n)\sin\omega(n) \right\}$$

Where

$$c(n) = 2h\left(\frac{N-1}{2} - n\right)$$

From this we can express the amplitude and phase function

Amplitude

$$|H(e^{j\omega})| = \sum_{n=1}^{\frac{N-1}{2}} c(n)\cos\omega(n)$$

Phase

$$\angle H(e^{j\omega}) = \frac{\pi}{2} - \left(\frac{N-1}{2}\right)\omega = \frac{\pi}{2} - \alpha\omega$$

Case 4 : Ant symmetrical impulse response and N - even

In the similar way of symmetric, we get for antisymmetric as

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} e^{\frac{j\pi}{2}} \left\{ \sum_{n=1}^{\frac{N}{2}} d(n) \sin \omega \left(n - \frac{1}{2}\right) \right\}$$

Where

$$d(n) = 2h\left(\frac{N}{2} - n\right)$$

From this we can express the amplitude and phase function

Amplitude

$$|H(e^{j\omega})| = \sum_{n=1}^{\frac{N}{2}} d(n) \sin \omega \left(n - \frac{1}{2}\right)$$

Phase

$$\angle H(e^{j\omega}) = \frac{\pi}{2} - \left(\frac{N-1}{2}\right)\omega = \frac{\pi}{2} - \alpha\omega$$

SUMMARY

Case1 : Symmetrical impulse response and N - odd

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=1}^{\frac{N-1}{2}} a(n)\cos\omega n\right\}$$

Where

$$a(0) = h\left(\frac{N-1}{2}\right) \quad a(n) = 2h\left(\frac{N-1}{2} - n\right)$$

Case2 : Symmetrical impulse response and N – even

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}\left\{\sum_{n=1}^{\frac{N}{2}} b(n)\cos\omega\left(n - \frac{1}{2}\right)\right\}$$

Where

$$b(n) = 2h\left(\frac{N}{2} - n\right)$$

Case 3 : Ant symmetrical impulse response and N - odd

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}e^{\frac{j\pi}{2}}\left\{\sum_{n=1}^{\frac{N-1}{2}} c(n)\sin\omega(n)\right\}$$

Where

$$c(n) = 2h\left(\frac{N-1}{2} - n\right)$$

Case 4 : Ant symmetrical impulse response and N - even

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)}e^{\frac{j\pi}{2}}\left\{\sum_{n=1}^{\frac{N}{2}} d(n)\sin\omega\left(n - \frac{1}{2}\right)\right\}$$

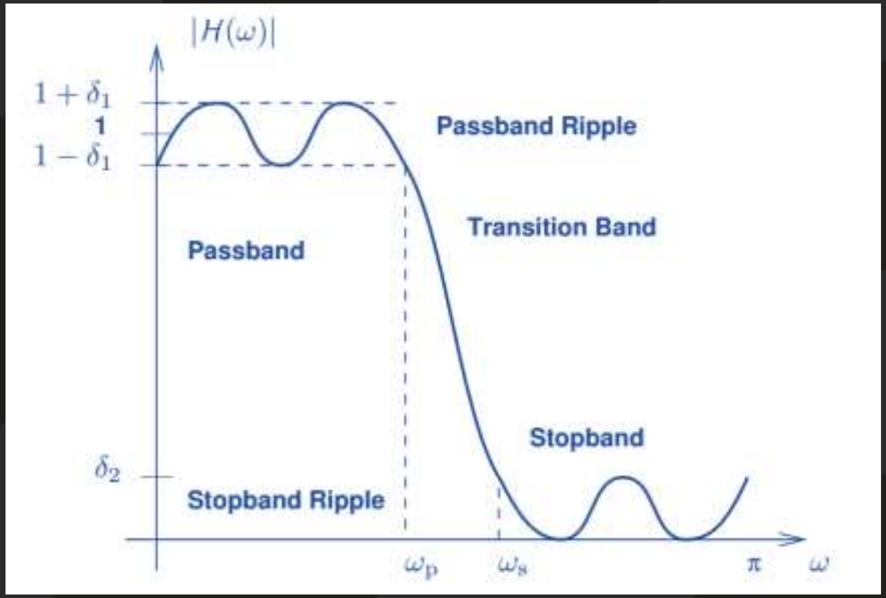
Where

$$d(n) = 2h\left(\frac{N}{2} - n\right)$$

Design of linear phase FIR filter

Why do we need a filter?

Frequency response of a practical lowpass filter



Digital filter design

1. ***Determining specification*** *: we need to know how strong the noise component is relative to the desired signal and how much we need to suppress the noise. This information is necessary to find the filter with minimum order for this application.*
2. ***Finding a transfer function*** *: we need to find a transfer function H(z) which will provide the required filtering.*
3. ***Choosing a realization structure*** *: there are many systems which can give the obtained transfer function and we must choose the appropriate one.*
4. ***Implementing the filter*** *: You have a couple of options for this step: a software implementation (such as a MATLAB or C code) or a hardware implementation (such as a DSP, a microcontroller, or an ASIC).*

Design of linear phase FIR filter : window method

Suppose that we want to design a lowpass filter with a cut off frequency of ω_c ,given frequency response

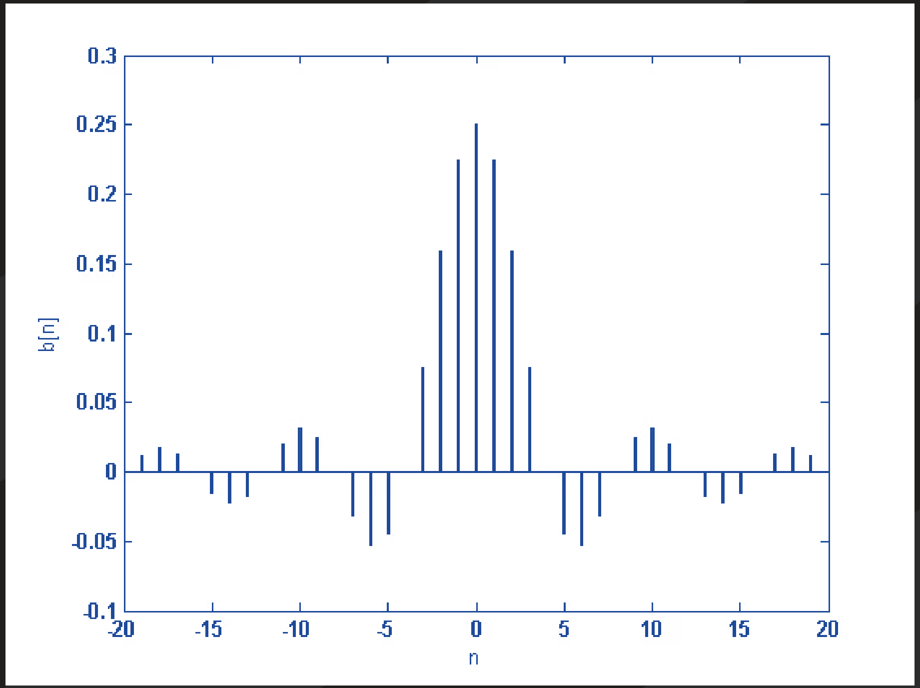
$$H_d(\omega) = \begin{cases} 1 \ , & |\omega| < \omega_c \\ 0 \ , & otherwise \end{cases}$$

To find the equivalent time-domain representation, we calculate the inverse discrete-time Fourier transform

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}d\omega$$

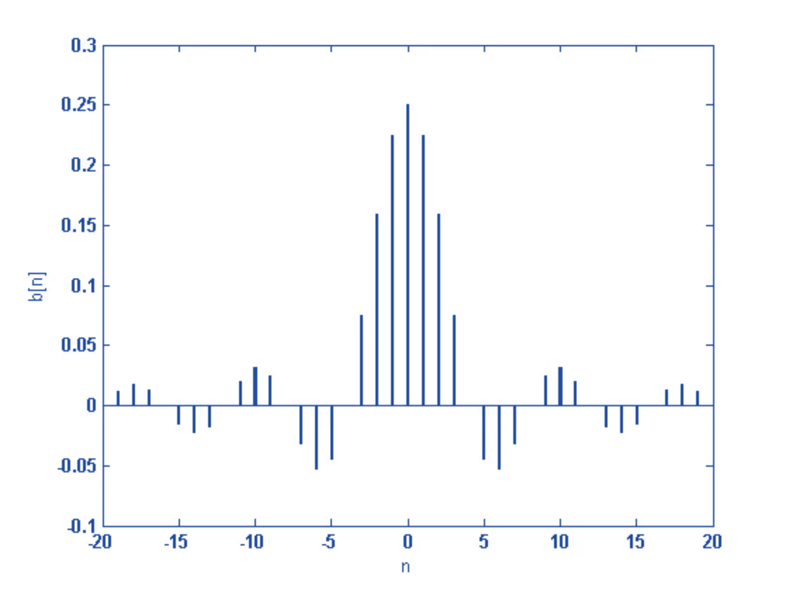
$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$$

$$= \frac{\sin(n\omega_c)}{n\pi}$$

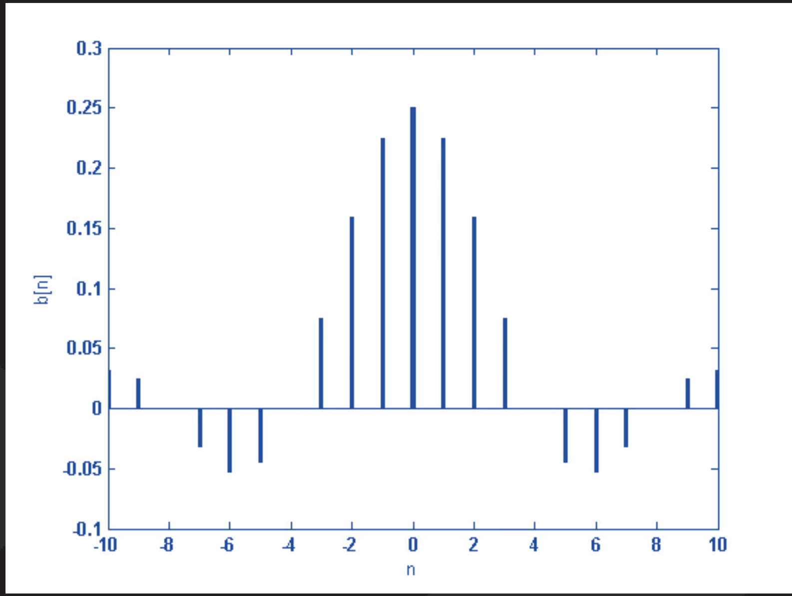


needs an infinite number of input samples to perform filtering and that the system is not a causal system. The solution will be to truncate the impulse response and use,

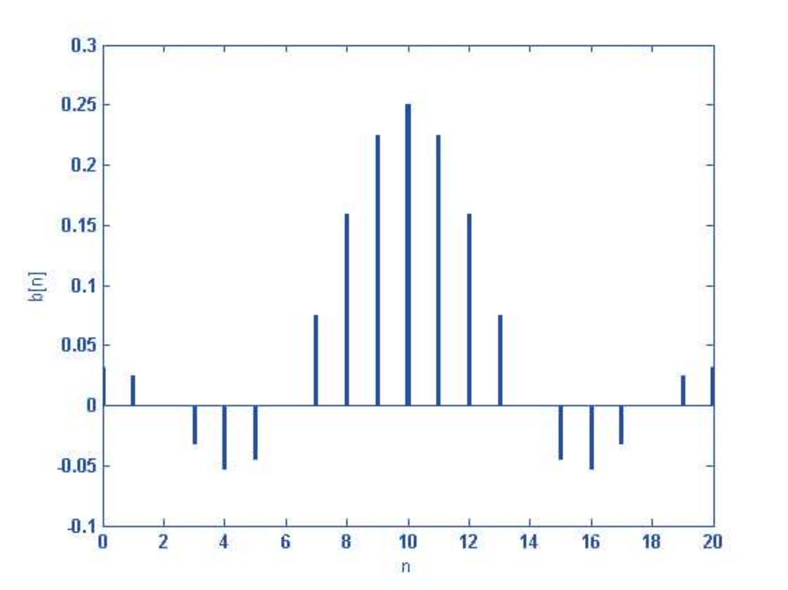
Design of linear phase FIR filter : window method



$h_d(n)$



Non causal system



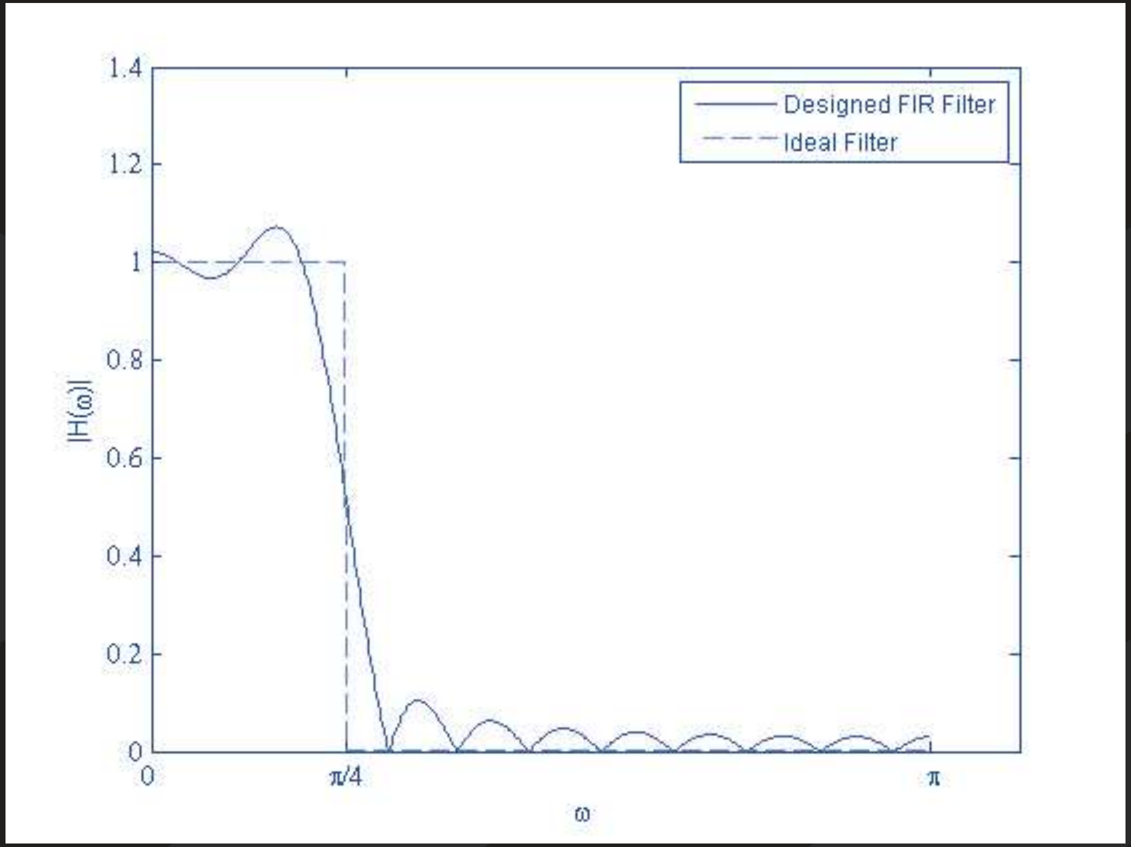
causal system & linear

but the system is delayed by $n = \frac{N-1}{2}$

There for considering an applied shift to $h_d(n)$ *and then multiplying with window function* $W(n)$

$$h(n) = h_d\left(n - \frac{N-1}{2}\right) * W(n)$$

Design of linear phase FIR filter : window method



Frequency response of the filter designed by a rectangular window

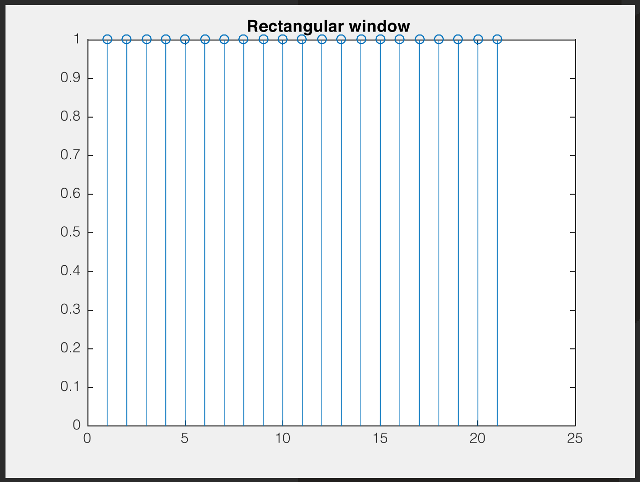
Design of linear phase FIR filter : window methods

Rectangular window

$$W_R(n) = \begin{cases} 1 & \frac{-(N-1)}{2} \leq n \leq \frac{(N-1)}{2} \\ 0 & otherwise \end{cases}$$

or

$$W_R(n) = \begin{cases} 1 & 0 \leq n \leq N \\ 0 & otherwise \end{cases}$$

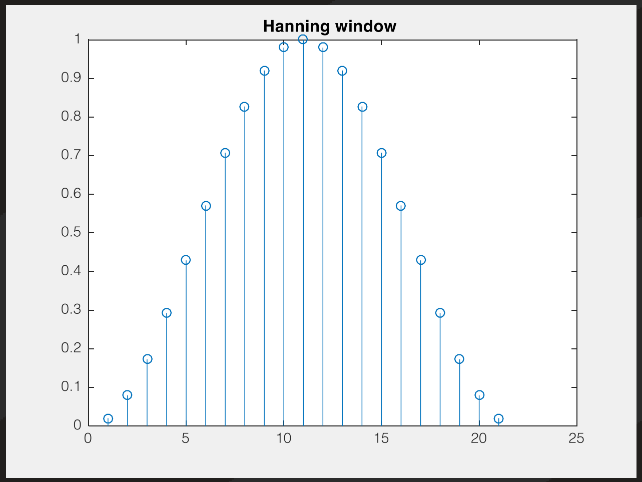


Hanning window

$$W_{Hn}(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right), & \frac{-(N-1)}{2} \leq n \leq \frac{(N-1)}{2} \\ 0\ , & otherwise \end{cases}$$

or

$$W_R(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N}\right), & 0 \leq n \leq N \\ 0\ , & otherwise \end{cases}$$

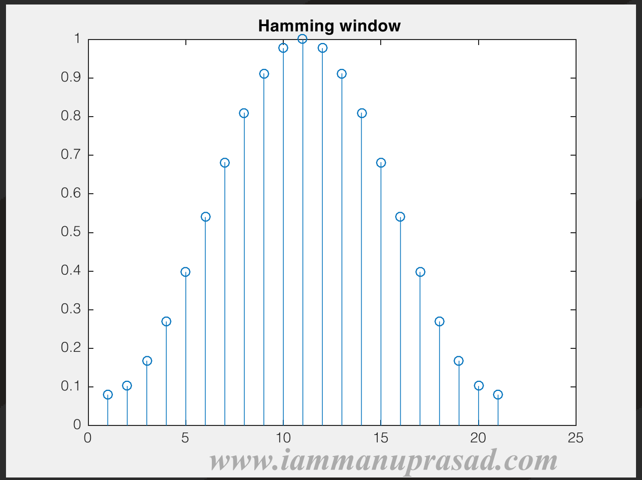


Hamming window

$$W_{Hm}(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), & \frac{-(N-1)}{2} \leq n \leq \frac{(N-1)}{2} \\ 0\ , & otherwise \end{cases}$$

or

$$W_R(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N}\right), & 0 \leq n \leq N \\ 0\ , & otherwise \end{cases}$$



Design of linear phase FIR filter : window method

Design procedure

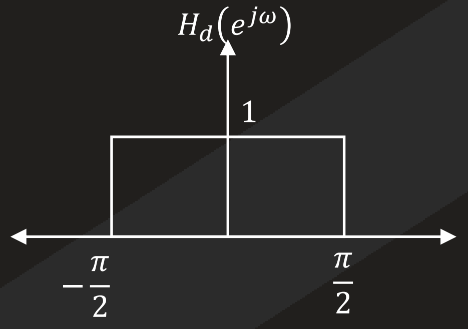
1. *Choose desired frequency response of the filter* $H_d(e^{j\omega})$
2. *Take the invert Fourier transform of* $H_d(e^{j\omega})$ *to obtain* $h_d(n)$
3. *Choose a window sequence* $W(n)$ *and multiply it with* $h_d(n)$ *to convert infinite duration impulse response to finite duration impulse response*

   $$h(n) = h_d(n) * W(n)$$

4. *The transfer function of the filter is obtained by taking Z-transform of h(n)*

Q) Design an ideal lowpass filter with frequency response $H_d(e^{j\omega}) = \begin{cases} 1 & for -\frac{\pi}{2} \le \omega \le \frac{\pi}{2} \\ 0 & otherwise \end{cases}$

find the value of h(n) for N=11 find H(z).



From the figure we know $\alpha = 0$

Solution

We can determine the desired impulse response $h_d(n)$ by taking inverse Fourier Transform

$$h_d(n) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi jn}\left[e^{\frac{jn\pi}{2}} - e^{\frac{-jn\pi}{2}}\right]$$

$$= \frac{1}{2\pi jn}\left[2j\sin\frac{\pi n}{2}\right] \qquad = \frac{\sin\frac{\pi n}{2}}{n\pi}$$

Truncating $h_d(n)$ to 11 samples

$$h_d(n) = \begin{cases} \frac{\sin\frac{\pi n}{2}}{n\pi} & for = -5 \le n \le 5 \\ 0 & otherwise \end{cases}$$

Since for n=0 the equation becomes infinity so lets apply limit

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\lim_{n\to 0}\frac{\sin n}{n} = 1$$

for n=0

$$h(0) = \lim_{n\to 0}\frac{\sin\frac{\pi n}{2}}{n\pi} = \lim_{n\to 0}\frac{\sin\frac{\pi n}{2}}{\frac{n\pi}{2}2} = \frac{1}{2}$$

for n=1

$$h(1) = \frac{\sin\frac{\pi}{2}}{\pi} = \frac{1}{\pi} = 0.318 = h(-1)$$

for n=2

$$h(2) = \frac{\sin\pi}{2\pi} = 0 = h(-2)$$

for n=3

$$h(3) = \frac{\sin\frac{3\pi}{2}}{3\pi} = \frac{-1}{3\pi} = -0.106 = h(-3)$$

for n=4

$$h(4) = \frac{\sin\frac{4\pi}{2}}{4\pi} = 0 = h(-4)$$

for n=5

$$h(5) = \frac{\sin\frac{5\pi}{2}}{5\pi} = \frac{1}{5\pi} = 0.0636 = h(-5)$$

Now lets find the transfer function of the filter by taking Z Transform

$$H(Z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n)Z^{-n} \quad = \sum_{n=-5}^{5} h(n)Z^{-n}$$

$$= h(-5)Z^{5} + h(-4)Z^{4} + h(-3)Z^{3} + h(-2)Z^{2} + h(-1)Z^{1} + h(0) + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + h(4)Z^{-4} + h(5)Z^{-5}$$

$$= h(0) + \sum_{n=1}^{5} h(n)[Z^{n} + Z^{-n}]$$

$$= 0.5 + 0.318(Z^{1} + Z^{-1}) + 0 - 0.106(Z^{3} + Z^{-3}) + 0 + 0.0636(Z^{5} + Z^{-5})$$

The transfer function of the realizable filter is

$$H'(Z) = Z^{-\left(\frac{N-1}{2}\right)} H(Z)$$

$$= Z^{-5}[0.5 + 0.318(Z^{1} + Z^{-1}) - 0.106(Z^{3} + Z^{-3}) + 0.0636(Z^{5} + Z^{-5})]$$

$$H'(Z) = 0.0636 - 0.106Z^{-2} + 0.318Z^{-4} + 0.5Z^{-5} + 0.318Z^{-6} - 0.106Z^{-8} + 0.0636Z^{-10}$$

$h(0) = h(10) = 0.0636$

$h(1) = h(9) = 0$

$h(2) = h(8) = -0.106$

$h(3) = h(7) = 0$

$h(4) = h(6) = 0.318$

$h(5) = 0$

Design of linear phase FIR filter : window method

Design Steps

1. *Plot the desired frequency response* $H_d(e^{j\omega})$
2. *Determine the desired impulse response* $h_d(n)$ *by taking the inverse Fourier transform of* $H_d(e^{j\omega})$

$$\boldsymbol{h_d(n) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.\, e^{j\omega n} d\omega}$$

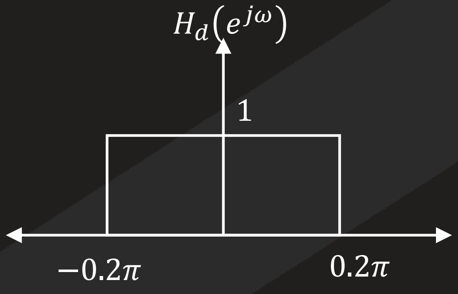
3. *Find the value of* $h_d(n)$ *for all 'n'*
4. *Choose a window sequence* $W(n)$ *and multiply it with* $h_d(n)$ *to get impulse response h(n)*

$$h(n) = h_d(n) * W(n)$$

5. *Take the Z – Transform of h(n)to get transfer function of the filter which is given and find coefficients*

$$H'(Z) = Z^{-\left(\frac{N-1}{2}\right)} H(Z)$$

FIR FILTER DESIGN USING RECTANGULAR WINDOW



Q) Design a linear phase FIR low pass filter using rectangular window by taking 7 samples of window sequence and with a cut off frequency $\omega_c = 0.2\pi \; rad/sec$

or

Q) Design a linear phase FIR filter low pass filter with frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & for - \omega_c \leq \omega \leq \omega \\ 0 & otherwise \end{cases}$$ where $\omega_c = 0.2\pi$ and N=7

<u>*Solution*</u>

We can determine the desired impulse response $h_d(n)$ by taking inverse Fourier Transform

$$h_d(n) = \frac{1}{2\pi}\int_{-0.2\pi}^{0.2\pi} 1.e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi jn}\left[e^{j0.2\pi n} - e^{-j0.2\pi n}\right]$$

$$= \frac{1}{2\pi jn}[2j \sin 0.2\pi n] \quad = \frac{\sin 0.2\pi n}{n\pi}$$

Truncating $h_d(n)$ to 7 samples

$$h_d(n) = \begin{cases} \frac{\sin 0.2\pi n}{n\pi} & for = -3 \leq n \leq 3 \\ 0 & otherwise \end{cases}$$

Since for n=0 the equation becomes infinity so lets apply limit

<u>*for n=0*</u>

$$h_d(0) = \lim_{n\to 0}\frac{\sin 0.2\pi n}{n\pi} = \lim_{n\to 0}\frac{\sin 0.2\pi n}{\frac{n\pi}{0.2}0.2} \quad = 0.2$$

<u>*for n=1*</u>

$$h_d(1) = \frac{\sin 0.2\pi}{\pi} \quad = 0.187 \quad = h_d(-1)$$

<u>*for n=2*</u>

$$h_d(2) = \frac{\sin 0.2\pi 2}{2\pi} \quad = 0.151 \quad = h_d(-2)$$

<u>*for n=3*</u>

$$h_d(3) = \frac{\sin 0.2\pi 3}{3\pi} \quad = 0.1009 \quad = h_d(-3)$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\lim_{n\to 0}\frac{\sin n}{n} = 1$$

FIR FILTER DESIGN USING RECTANGULAR WINDOW

Now using rectangular window sequence W(n) and multiply $h_d(n)$ with it to get the impulse response *h(n)*

Rectangular window

$$W_R(n) = \begin{cases} 1 & \frac{-(N-1)}{2} \le n \le \frac{(N-1)}{2} \\ 0 & otherwise \end{cases}$$

$$W_R(n) = \begin{cases} 1 & 0 \le n \le 3 \\ 0 & otherwise \end{cases}$$

Since $\alpha = 0$ *we get a non causal filter coefficient symmetrical about n=0 so h(n) = h(-n)*

for n=0 $\quad h(0) = h_d(0).W_R(0) \quad = 0.2$

for n=1 $\quad h(1) = h_d(1).W_R(1) \quad = 0.187 = h(-1)$

for n=2 $\quad h(2) = h_d(2).W_R(2) \quad = 0.1514 = h(-2)$

for n=3 $\quad h(3) = h_d(3).W_R(3) \quad = 0.1009 = h(-3)$

$$h(n) = [0.1009, 0.1514, 0.187, 0.2, 0.187, 0.1514, 0.1009]$$

Now lets find the transfer function of the filter by taking Z Transform

$$H(Z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n)Z^{-n} \quad = \sum_{n=-3}^{3} h(n)Z^{-n}$$

$$= h(-3)Z^3 + h(-2)Z^2 + h(-1)Z^1 + h(0) + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3}$$

$$H(Z) = h(0) + \sum_{n=1}^{3} h(n)[Z^n + Z^{-n}]$$

$$= 0.2 + 0.187(Z^1 + Z^{-1}) + 0.151(Z^2 + Z^{-2}) + 0.1009(Z^3 + Z^{-3})$$

The transfer function of the realizable filter is

$$H'(Z) = Z^{-\left(\frac{N-1}{2}\right)} H(Z)$$

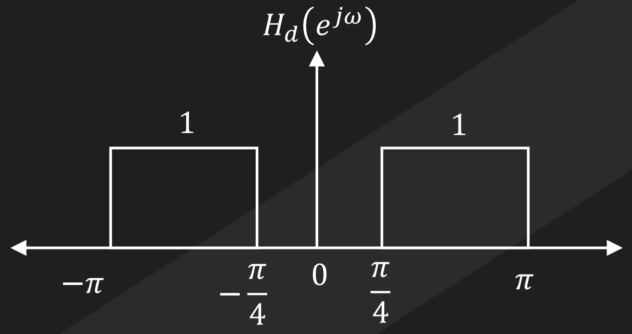
$$= Z^{-3}[0.2 + 0.187(Z^1 + Z^{-1}) + 0.151(Z^2 + Z^{-2}) + 0.1009(Z^3 + Z^{-3})]$$

$$= 0.1009 + 0.151Z^{-1} + 0.187Z^{-2} + 0.2Z^{-3} + 0.187Z^{-4} + 0.151Z^{-5} + 0.1009Z^{-6}$$

$h(0) = h(6) = 0.1009$ $\qquad h(1) = h(5) = 0.151$

$h(2) = h(4) = 0.187$ $\qquad h(3) = 0.2$

FIR FILTER DESIGN USING RECTANGULAR WINDOW



Q) Design a linear phase FIR high pass filter with frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & for\ \frac{\pi}{4} \le |\omega| \le \pi \\ 0 & for\ |\omega| < \frac{\pi}{4} \end{cases}$$

Find the value of *h(n)* for N=11 and find H(z). Use rectangular window

From the figure we know $\alpha = 0$ also symmetric

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\lim_{n\to 0} \frac{\sin n}{n} = 1$$

Solution

We can determine the desired impulse response $h_d(n)$ by taking inverse Fourier Transform

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}).e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\frac{\pi}{4}} 1.e^{j\omega n} d\omega + \int_{\frac{\pi}{4}}^{\pi} 1.e^{j\omega n} d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\frac{\pi}{4}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\frac{\pi}{4}}^{\pi}$$

$$= \frac{-1}{2\pi jn}\left[e^{\frac{jn\pi}{4}} - e^{\frac{-jn\pi}{4}} - (e^{j\pi n} - e^{-j\pi n})\right]$$

$$= \frac{-1}{2\pi jn}\left[2j\sin\frac{n\pi}{4} - 2j\sin n\pi\right]$$

$$h_d(n) = \frac{\sin n\pi - \sin\frac{n\pi}{4}}{\pi n}$$

Truncating $h_d(n)$ to 11 samples

Since for n=0 the equation becomes infinity so lets apply limit

for n=0

$$h_d(0) = \lim_{n\to 0}\frac{\sin \pi n}{n\pi} - \lim_{n\to 0}\frac{\sin\frac{\pi n}{4}}{\frac{n\pi}{4}4} = 1 - \frac{1}{4} = \frac{3}{4}$$

for n=1

$$h_d(1) = \frac{\sin\pi - \sin\frac{\pi}{4}}{\pi} = -0.225 = h_d(-1)$$

for n=2

$$h_d(2) = \frac{\sin 2\pi - \sin\frac{2\pi}{4}}{2\pi} = -0.1591 = h_d(-2)$$

for n=3

$$h_d(3) = \frac{\sin 3\pi - \sin\frac{3\pi}{4}}{3\pi} = -0.075 = h_d(-3)$$

for n=4

$$h_d(4) = \frac{\sin 4\pi - \sin\frac{4\pi}{4}}{4\pi} = 0 = h_d(-4)$$

for n=5

$$h_d(5) = \frac{\sin 5\pi - \sin\frac{5\pi}{4}}{5\pi} = 0.045 = h_d(-5)$$

FIR FILTER DESIGN USING RECTANGULAR WINDOW

Now using rectangular window sequence $W_R(n)$ and multiply $h_d(n)$ with it to get the impulse response *h(n)*

$$W_R(n) = \begin{cases} 1 & 0 \le n \le 5 \\ 0 & otherwise \end{cases}$$

Since $\alpha = 0$ *we get a non causal filter coefficient symmetrical about n=0 so h(n) = h(-n)*

Rectangular window

$$W_R(n) = \begin{cases} 1 & \frac{-(N-1)}{2} \le n \le \frac{(N-1)}{2} \\ 0 & otherwise \end{cases}$$

for n=0 $h(0) = h_d(0).W_R(0) = 0.75$

for n=1 $h(1) = h_d(1).W_R(1) = -0.225 = h(-1)$

for n=2 $h(2) = h_d(2).W_R(2) = -0.1591 = h(-2)$

for n=3 $h(3) = h_d(3).W_R(3) = -0.075 = h(-3)$

for n=4 $h(4) = h_d(4).W_R(4) = 0 = h(-4)$

for n=5 $h(5) = h_d(5).W_R(5) = 0.0450 = h(-5)$

$$h(n) = [0.0450, 0, -0.075, -0.1591, -0.225, 0.75, -0.225 - 0.1591 - 0.0750, 0.0450]$$

Now lets find the transfer function of the filter by taking Z Transform

$$H(Z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n)Z^{-n} = \sum_{n=-5}^{5} h(n)Z^{-n}$$

$$= h(-5)Z^5 + h(-4)Z^4 + h(-3)Z^3 + h(-2)Z^2 + h(-1)Z^1 + h(0) + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + h(4)Z^{-4} + h(5)Z^{-5}$$

$$H(Z) = h(0) + \sum_{n=1}^{3} h(n)[Z^n + Z^{-n}]$$

$$= 0.75 - 0.225(Z^1 + Z^{-1}) - 0.159(Z^2 + Z^{-2}) - 0.075(Z^3 + Z^{-3}) + 0.045(Z^5 + Z^{-5})$$

The transfer function of the realizable filter is

$$H'(Z) = Z^{-\left(\frac{N-1}{2}\right)} H(Z)$$

$$= Z^{-5}[0.75 - 0.225(Z^1 + Z^{-1}) - 0.159(Z^2 + Z^{-2}) - 0.075(Z^3 + Z^{-3}) + 0.045(Z^5 + Z^{-5})]$$

$$= 0.045 - 0.075Z^{-2} - 0.159Z^{-3} - 0.225Z^{-4} + 0.75Z^{-5} - 0.225Z^{-6} - 0.1591Z^{-7} - 0.075Z^{-8} + 0.045Z^{-10}$$

$h(0) = h(10) = 0.045$ $\quad h(3) = h(7) = -0.159$

$h(1) = h(9) = 0$ $\quad h(4) = h(6) = -0.225$

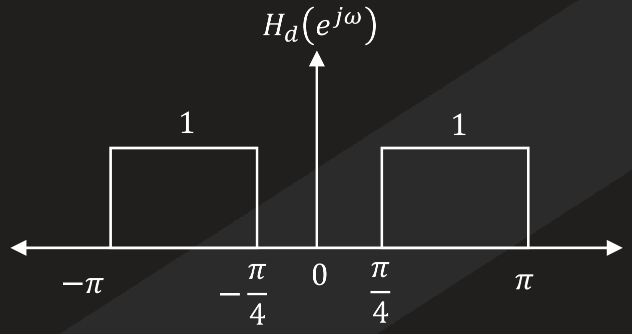
$h(2) = h(8) = -0.075$ $\quad h(5) = 0.75$

FIR FILTER DESIGN USING HANNING WINDOW

Q) Design a linear phase FIR filter high pass filter with frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & for\ \frac{\pi}{4} \le |\omega| \le \pi \\ 0 & for\ |\omega| < \frac{\pi}{4} \end{cases}$$

Find the value of *h(n)* for N=11 and find H(z). Using Hanning window



From the figure we know $\alpha = 0$ also symmetric

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\lim_{n\to 0} \frac{\sin n}{n} = 1$$

Solution

We can determine the desired impulse response $h_d(n)$ by taking inverse Fourier Transform

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}).e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\frac{\pi}{4}} 1.e^{j\omega n} d\omega + \int_{\frac{\pi}{4}}^{\pi} 1.e^{j\omega n} d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\frac{\pi}{4}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\frac{\pi}{4}}^{\pi}$$

$$= \frac{-1}{2\pi jn}\left[e^{\frac{jn\pi}{4}} - e^{\frac{-jn\pi}{4}} - \left(e^{j\pi n} - e^{-j\pi n}\right)\right]$$

$$= \frac{-1}{2\pi jn}\left[2j\sin\frac{n\pi}{4} - 2j\sin n\pi\right]$$

$$h_d(n) = \frac{\sin n\pi - \sin\frac{n\pi}{4}}{\pi n}$$

Truncating $h_d(n)$ to 11 samples

Since for n=0 the equation becomes infinity so lets apply limit

for n=0

$$h_d(0) = \lim_{n\to 0}\frac{\sin \pi n}{n\pi} - \lim_{n\to 0}\frac{\sin\frac{\pi n}{4}}{\frac{n\pi}{4}\,4} = 1 - \frac{1}{4} = \frac{3}{4}$$

for n=1

$$h_d(1) = \frac{\sin\pi - \sin\frac{\pi}{4}}{\pi} = -0.225 = h_d(-1)$$

for n=2

$$h_d(2) = \frac{\sin 2\pi - \sin\frac{2\pi}{4}}{2\pi} = -0.1591 = h_d(-2)$$

for n=3

$$h_d(3) = \frac{\sin 3\pi - \sin\frac{3\pi}{4}}{3\pi} = -0.075 = h_d(-3)$$

for n=4

$$h_d(4) = \frac{\sin 4\pi - \sin\frac{4\pi}{4}}{4\pi} = 0 = h_d(-4)$$

for n=5

$$h_d(5) = \frac{\sin 5\pi - \sin\frac{5\pi}{4}}{5\pi} = 0.045 = h_d(-5)$$

Hanning window

$$W_{Hn}(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right), & \frac{-(N-1)}{2} \le n \le \frac{(N-1)}{2} \\ 0\ , & otherwise \end{cases}$$

Since $\alpha = 0$ *we get a non causal filter coefficient symmetrical about n=0 so h(n) = h(-n)*

$$W_{Hn}(n) = \begin{cases} 0.5 + 0.5\cos\frac{2n\pi}{10} & 0 \le n \le 5 \\ 0 & otherwise \end{cases}$$

$$W_{Hn}(0) = 0.5 + 0.5 = 1$$

$$W_{Hn}(1) = 0.5 + 0.5\cos\frac{\pi}{5} = 09045 = W_{Hn}(-1)$$

$$W_{Hn}(2) = 0.5 + 0.5\cos\frac{2\pi}{5} = 0.655 = W_{Hn}(-2)$$

$$W_{Hn}(3) = 0.5 + 0.5\cos\frac{3\pi}{5} = 0.345 = W_{Hn}(-3)$$

$$W_{Hn}(4) = 0.5 + 0.5\cos\frac{4\pi}{5} = 0.0945 = W_{Hn}(-4)$$

$$W_{Hn}(5) = 0.5 + 0.5\cos\frac{5\pi}{5} = 0 = W_{Hn}(-5)$$

FIR FILTER DESIGN USING HANNING WINDOW

Now using Hanning window sequence $W_{Hn}(n)$ and multiply $h_d(n)$ with it to get the impulse response *h(n)*

$$h(n) = h_d(n).W_{Hn}(n) \ \ for\ -5 \le n \le 5$$

for n=0 $\quad h(0) = h_d(0).W_{Hn}(0) = 0.75(1) = 0.75$

for n=1 $\quad h(1) = h_d(1).W_{Hn}(1) = -0.225(0.905) = -0.204 = h(-1)$

for n=2 $\quad h(2) = h_d(2).W_{Hn}(2) = -0.159(0.655) = -0.104 = h(-2)$

for n=3 $\quad h(3) = h_d(3).W_{Hn}(3) = -0.075(0.345) = -0.026 = h(-3)$

for n=4 $\quad h(4) = h_d(4).W_{Hn}(4) = 0 = h(-4)$

for n=5 $\quad h(5) = h_d(5).W_{Hn}(5) = 0 = h(-5)$

$$h(n) = [-0.026, -0.104, -0.204, 0.75, -0.204, -0.104, -0.026]$$

Now lets find the transfer function of the filter by taking Z Transform

$$H(Z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n)Z^{-n} = \sum_{n=-5}^{5} h(n)Z^{-n}$$

$$= h(-5)Z^5 + h(-4)Z^4 + h(-3)Z^3 + h(-2)Z^2 + h(-1)Z^1 + h(0) + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + h(4)Z^{-4} + h(5)Z^{-5}$$

$$H(Z) = h(0) + \sum_{n=1}^{5} h(n)[Z^n + Z^{-n}]$$

$$= 0.75 - 0.204(Z^1 + Z^{-1}) - 0.104(Z^2 + Z^{-2}) - 0.026(Z^3 + Z^{-3})$$

The transfer function of the realizable filter is

$$H'(Z) = Z^{-\left(\frac{N-1}{2}\right)} H(Z)$$

$$= Z^{-5}[0.75 - 0.204(Z^1 + Z^{-1}) - 0.104(Z^2 + Z^{-2}) - 0.026(Z^3 + Z^{-3})]$$

$$= -0.026Z^{-2} - 0.104Z^{-3} - 0.204Z^{-4} + 0.75Z^{-5} - 0.204Z^{-6} - 0.104Z^{-7} - 0.026Z^{-8}$$

$$h(0) = h(1) = h(9) = h(10) = 0$$

$h(2) = h(8) = -0.026$ $\qquad h(4) = h(6) = -0.204$

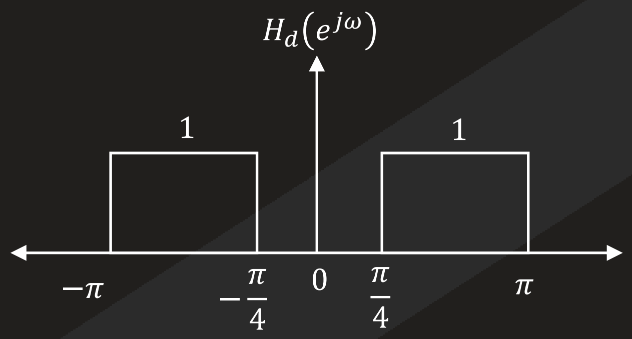
$h(3) = h(7) = -0.104$ $\qquad h(5) = 0.75$

FIR FILTER DESIGN USING HAMMING WINDOW

Q) Design a linear phase FIR filter high pass filter with frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & for\ \frac{\pi}{4} \le |\omega| \le \pi \\ 0 & for\ |\omega| < \frac{\pi}{4} \end{cases}$$

Find the value of *h(n)* for N=11 and find H(z). Using Hamming window



From the figure we know $\alpha = 0$
also symmetric

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\lim_{n\to 0} \frac{\sin n}{n} = 1$$

Solution

We can determine the desired impulse response $h_d(n)$ *by taking inverse Fourier Transform*

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}).e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\frac{\pi}{4}} 1.e^{j\omega n}d\omega + \int_{\frac{\pi}{4}}^{\pi} 1.e^{j\omega n}d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi}^{-\frac{\pi}{4}} + \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{\frac{\pi}{4}}^{\pi}$$

$$= \frac{-1}{2\pi jn}\left[e^{\frac{jn\pi}{4}} - e^{\frac{-jn\pi}{4}} - \left(e^{j\pi n} - e^{-j\pi n}\right)\right]$$

$$= \frac{-1}{2\pi jn}\left[2j\sin\frac{n\pi}{4} - 2j\sin n\pi\right]$$

$$h_d(n) = \frac{\sin n\pi - \sin\frac{n\pi}{4}}{\pi n}$$

Truncating $h_d(n)$ *to 11 samples*

Since for n=0 the equation becomes infinity so lets apply limit

for n=0

$$h_d(0) = \lim_{n\to 0}\frac{\sin \pi n}{n\pi} - \lim_{n\to 0}\frac{\sin\frac{\pi n}{4}}{\frac{n\pi}{4}4} = 1 - \frac{1}{4} = \frac{3}{4}$$

for n=1

$$h_d(1) = \frac{\sin\pi - \sin\frac{\pi}{4}}{\pi} = -0.225 = h_d(-1)$$

for n=2

$$h_d(2) = \frac{\sin 2\pi - \sin\frac{2\pi}{4}}{2\pi} = -0.1591 = h_d(-2)$$

for n=3

$$h_d(3) = \frac{\sin 3\pi - \sin\frac{3\pi}{4}}{3\pi} = -0.075 = h_d(-3)$$

for n=4

$$h_d(4) = \frac{\sin 4\pi - \sin\frac{4\pi}{4}}{4\pi} = 0 = h_d(-4)$$

for n=5

$$h_d(5) = \frac{\sin 5\pi - \sin\frac{5\pi}{4}}{5\pi} = 0.045 = h_d(-5)$$

FIR FILTER DESIGN USING HAMMING WINDOW

Hamming window

$$W_H(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), & \frac{-(N-1)}{2} \le n \le \frac{(N-1)}{2} \\ 0, & otherwise \end{cases}$$

Since $\alpha = 0$ *we get anon causal filter coefficient symmetrical about n=0 so h(n) = h(-n)*

$$W_H(n) = \begin{cases} 0.54 + 0.46\cos\frac{2n\pi}{10} & 0 \le n \le 5 \\ 0 & otherwise \end{cases}$$

$$W_H(0) = 0.54 + 0.46 = 1$$

$$W_H(1) = 0.54 + 0.46\cos\frac{\pi}{5} = 0.912 = W_H(-1)$$

$$W_H(2) = 0.54 + 0.46\cos\frac{2\pi}{5} = 0.682 = W_H(-2)$$

$$W_H(3) = 0.54 + 0.46\cos\frac{3\pi}{5} = 0.398 = W_H(-3)$$

$$W_H(4) = 0.54 + 0.46\cos\frac{4\pi}{5} = 0.1678 = W_H(-4)$$

$$W_H(5) = 0.54 + 0.46\cos\frac{5\pi}{5} = 0.08 = W_H(-5)$$

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FIR FILTER DESIGN USING HAMMING WINDOW

Now using Hamming window sequence $W_H(n)$ and multiply $h_d(n)$ with it to get the impulse response *h(n)*

$$h(n) = h_d(n).W_H(n) \;\; for \; -5 \le n \le 5$$

for n=0 $\quad h(0) = h_d(0).W_H(0) = 0.75(1) = 0.75$

for n=1 $\quad h(1) = h_d(1).W_H(1) = -0.225(0.912) = -0.2056 = h(-1)$

for n=2 $\quad h(2) = h_d(2).W_H(2) = -0.159(0.682) = -0.1084 = h(-2)$

for n=3 $\quad h(3) = h_d(3).W_H(3) = -0.075(0.398) = -0.03 = h(-3)$

for n=4 $\quad h(4) = h_d(4).W_H(4) = 0 = h(-4)$

for n=5 $\quad h(5) = h_d(5).W_H(5) = -0.045(0.08) = 0.0036 = h(-5)$

$$h(n) = [0.0036, -0.03, -0.1084, -0.2056, 0.75, -0.2056, -0.1084, -0.03, 0.0036]$$

Now lets find the transfer function of the filter by taking Z Transform

$$H(Z) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\left(\frac{N-1}{2}\right)} h(n)Z^{-n} = \sum_{n=-5}^{5} h(n)Z^{-n}$$

$$= h(-5)Z^5 + h(-4)Z^4 + h(-3)Z^3 + h(-2)Z^2 + h(-1)Z^1 + h(0) + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + h(4)Z^{-4} + h(5)Z^{-5}$$

$$H(Z) = h(0) + \sum_{n=1}^{5} h(n)[Z^n + Z^{-n}]$$

$$= 0.75 - 0.2056(Z^1 + Z^{-1}) - 0.1084(Z^2 + Z^{-2}) - 0.03(Z^3 + Z^{-3}) + 0.0036(Z^5 + Z^{-5})$$

The transfer function of the realizable filter is

$$H'(Z) = Z^{-\left(\frac{N-1}{2}\right)}H(Z)$$

$$= Z^{-5}[0.75 - 0.2056(Z^1 + Z^{-1}) - 0.1084(Z^2 + Z^{-2}) - 0.03(Z^3 + Z^{-3}) + 0.0036(Z^5 + Z^{-5})]$$

$$= 0.0036 - 0.03Z^{-2} - 0.1084Z^{-3} - 0.2052Z^{-4} + 0.75Z^{-5} - 0.2052Z^{-6} - 0.1084Z^{-7} - 0.03Z^{-8} + 00036Z^{-10}$$

$h(0) = h(10) = 0.0036$ $\quad h(3) = h(7) = -0.1084$

$h(1) = h(9) = 0$ $\quad h(4) = h(6) = -0.2052$

$h(2) = h(8) = -0.03$ $\quad h(5) = 0.75$

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Design of linear phase FIR filter by frequency sampling technique

In this method the ideal frequency response is sampled at sufficient number of points these samples are the DFT coefficients of impulse response of filter. Hence impulse response of filter is determined by taking inverse DFT

Steps

1. *Choose a desired frequency response* $H_d\left(e^{j\omega}\right)$
2. *Sample* $H_d\left(e^{j\omega}\right)$ *at N point by taking* $\omega = \omega_k = \frac{2\pi k}{N}$ *where k=0,1,2,3 … N-1*
3. *Compute the N samples of impulse response h(n) using the equation*

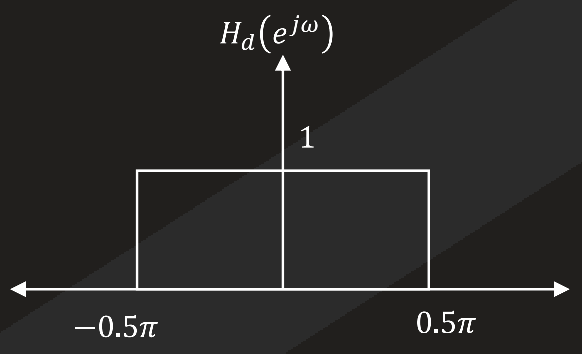
$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k)e^{\frac{j2\pi nk}{N}}\right]\right], for\ N = odd$$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N}{2}-1} Re\left[H(k)e^{\frac{j2\pi nk}{N}}\right]\right], for\ N = even$$

4. *Take Z-Transform of impulse response h(n) to get filter transfer function H(z)*

$$H(Z) = \sum_{n=0}^{N-1} h(n)Z^{-n}$$

Design of linear phase FIR filter by frequency sampling technique

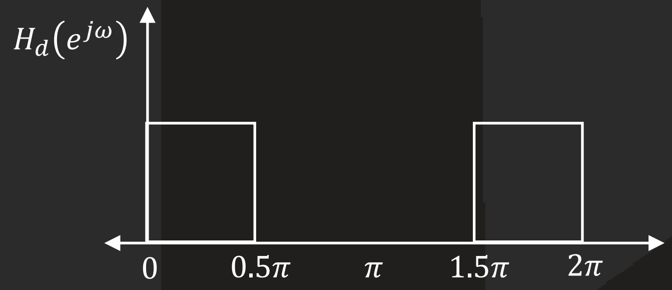


Q) Design a linear phase FIR low pass filter with cut off frequency of 0.5π rad/sample by taking 11 samples of ideal frequency response

Solution

Step1 : Find the desired frequency response

For digital sampling we are taking the limit as 0 to 2π



Due to symmetricity at (N-1)/2 then there will be an α exponential term in the expression

$$H_d(e^{j\omega}) = \begin{cases} 1.e^{-j\alpha\omega} & ,0 \le \omega \le 0.5\pi \\ 0 & ,0.5\pi \le \omega \le 1.5\pi \\ 1.e^{-j\alpha\omega} & ,1.5\pi \le \omega \le 2\pi \end{cases}$$

$$where, \alpha = \frac{N-1}{2} = \frac{11-1}{2} = 5$$

Step2: Sample $H_d(e^{j\omega})$ at N point by taking $\omega = \omega_k = \frac{2\pi k}{N}$

Sampling frequency $\omega_k = \frac{2\pi k}{11}$ *for k= 0 to 10*

for k=0	$\omega_0 = \frac{2\pi * 0}{11}$	$= 0$
for k=1	$\omega_0 = \frac{2\pi * 1}{11}$	$= 0.18\pi$
for k=2	$\omega_0 = \frac{2\pi * 2}{11}$	$= 0.36\pi$
for k=3	$\omega_0 = \frac{2\pi * 3}{11}$	$= 0.55\pi$
for k=4	$\omega_0 = \frac{2\pi * 4}{11}$	$= 0.73\pi$
for k=5	$\omega_0 = \frac{2\pi * 5}{11}$	$= 0.91\pi$
for k=6	$\omega_0 = \frac{2\pi * 6}{11}$	$= 1.09\pi$
for k=7	$\omega_0 = \frac{2\pi * 7}{11}$	$= 1.27\pi$
for k=8	$\omega_0 = \frac{2\pi * 8}{11}$	$= 1.45\pi$
for k=9	$\omega_0 = \frac{2\pi * 9}{11}$	$= 1.64\pi$
for k=10	$\omega_0 = \frac{2\pi * 10}{11}$	$= 1.82\pi$

$$H(k) = \begin{cases} e^{-j5\frac{2\pi k}{11}} & ,for\ k = 0,1,2 \\ 0 & ,for\ k = 3\ to\ 8 \\ e^{-j5\frac{2\pi k}{11}} & ,for\ k = 9,10 \end{cases}$$

Design of linear phase FIR filter by frequency sampling technique

Step 3: Compute the N samples of impulse response h(n)

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k)e^{\frac{j2\pi nk}{N}}\right]\right], for\ N = odd$$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k)e^{\frac{j2\pi nk}{N}}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\sum_{k=1}^{5} Re\left[e^{-j5\frac{2\pi k}{11}}e^{\frac{j2\pi nk}{11}}\right]\right] \qquad = \frac{1}{11}\left[1 + 2\sum_{k=1}^{2} Re\left[e^{-j5\frac{2\pi k}{11}}e^{\frac{j2\pi nk}{11}}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\sum_{k=1}^{2} Re\left[e^{j\frac{2\pi k}{11}(n-5)}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2Re\left[e^{j\frac{2\pi}{11}(n-5)}\right] + 2Re\left[e^{j\frac{4\pi}{11}(n-5)}\right]\right]$$

$$h(n) = \frac{1}{11}\left[1 + 2\cos\left(\frac{2\pi}{11}(n-5)\right) + 2\cos\left(\frac{4\pi}{11}(n-5)\right)\right]$$

$$H(k) = \begin{cases} e^{-j5\frac{2\pi k}{11}} & , for\ k = 0,1,2 \\ 0 & , for\ k = 3\ to\ 8 \\ e^{-j5\frac{2\pi k}{11}} & , for\ k = 9,10 \end{cases}$$

Design of linear phase FIR filter by frequency sampling technique

Now let's calculate h(n) for n = 0 to 10, using symmetric condition (h(n)=h(N-1-m))

for n=0

$$h(0) = \frac{1}{11}\left[1 + 2\cos\left(\frac{2\pi}{11}(0-5)\right) + 2\cos\left(\frac{4\pi}{11}(0-5)\right)\right] = 0.0694$$

for n=1

$$h(1) = \frac{1}{11}\left[1 + 2\cos\left(\frac{2\pi}{11}(1-5)\right) + 2\cos\left(\frac{4\pi}{11}(1-5)\right)\right] = -0.054$$

for n=2

$$h(2) = \frac{1}{11}\left[1 + 2\cos\left(\frac{2\pi}{11}(2-5)\right) + 2\cos\left(\frac{4\pi}{11}(2-5)\right)\right] = -0.1094$$

for n=3

$$h(3) = \frac{1}{11}\left[1 + 2\cos\left(\frac{2\pi}{11}(3-5)\right) + 2\cos\left(\frac{4\pi}{11}(3-5)\right)\right] = 0.0473$$

for n=4

$$h(4) = \frac{1}{11}\left[1 + 2\cos\left(\frac{2\pi}{11}(4-5)\right) + 2\cos\left(\frac{4\pi}{11}(4-5)\right)\right] = 0.3194$$

for n=5

$$h(5) = \frac{1}{11}\left[1 + 2\cos\left(\frac{2\pi}{11}(5-5)\right) + 2\cos\left(\frac{4\pi}{11}(-5)\right)\right] = 0.4595$$

for n=6

$$h(6) = h(11-1-6) = h(4) = 0.3194$$

for n=7

$$h(7) = h(11-1-7) = h(3) = 0.0473$$

for n=8

$$h(8) = h(11-1-8) = h(2) = -0.1094$$

for n=9

$$h(9) = h(11-1-9) = h(1) = -0.054$$

for n=10

$$h(10) = h(11-1-10) = h(0) = 0.0694$$

Design of linear phase FIR filter by frequency sampling technique

Step 4: Take Z-Transform of impulse response h(n) to get filter transfer function H(z)

$$H(Z) = \sum_{n=0}^{10} h(n)Z^{-n}$$

$$= h(0)Z^{0} + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + h(4)Z^{-4} + h(5)Z^{-5} + h(6)Z^{-6} + h(7)Z^{-7} + h(8)Z^{-8} + h(9)Z^{-9} + h(10)Z^{-10}$$

$$H(Z) = 0.0694(1 + Z^{-10}) - 0.054(Z^{-1} + Z^{-9}) - 0.1094(Z^{-2} + Z^{-8}) + 0.0473(Z^{-3} + Z^{-7}) + 0.3194(Z^{-4} + Z^{-6}) + 0.4595Z^{-5}$$

$h(6) = h(11 - 1 - 6) = h(4) \quad = 0.3194$

$h(7) = h(11 - 1 - 7) = h(3) \quad = 0.0473$

$h(8) = h(11 - 1 - 8) = h(2) \quad = -0.1094$

$h(9) = h(11 - 1 - 9) = h(1) \quad = -0.054$

$h(10) = h(11 - 1 - 10) = h(0) \quad = 0.0694$

Design of linear phase FIR filter by frequency sampling technique

Q) Using frequency sampling method, design a band pass filter with the following specifications, sampling frequency $F = 8000Hz$, cut off frequency $f_{c1} = 1000Hz$, $f_{c2} = 3000Hz$, Determine the filter coefficients for N=7

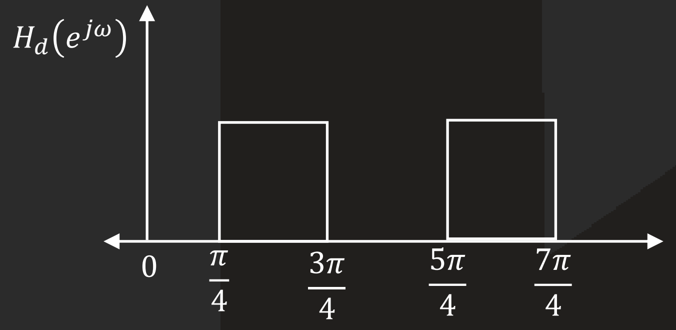
Solution

Step1 : Find the desired frequency response

For digital sampling we are taking the limit as 0 to 2π

$$\omega_{c1} = 2\pi f_{c1} T = \frac{2\pi f_{c1}}{F} = \frac{2\pi 1000}{8000} = \frac{\pi}{4} = 0.25\pi$$

$$\omega_{c2} = 2\pi f_{c2} T = \frac{2\pi f_{c2}}{F} = \frac{2\pi 3000}{8000} = \frac{3\pi}{4} = 0.75\pi$$



Due to symmetricity at (N-1)/2 then there will be an α exponential term in the expression

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & , 0.25\pi \le \omega \le 0.75\pi \\ 0 & , 0.75\pi \le \omega \le 1.25\pi \\ e^{-j\alpha\omega} & , 1.25\pi \le \omega \le 1.7\pi \end{cases}$$

$$where, \alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3$$

Step2: Sample $H_d(e^{j\omega})$ at N point by taking $\omega = \omega_k = \frac{2\pi k}{N}$

Sampling frequency $\omega_k = \frac{2\pi k}{7}$ *for k= 0 to 6*

for k=0 $\quad \omega_0 = \frac{2\pi * 0}{7} = 0$

for k=1 $\quad \omega_0 = \frac{2\pi * 1}{7} = 0.28\pi$

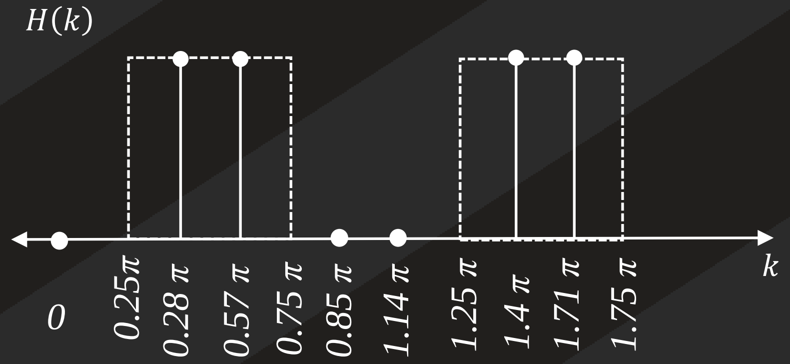
for k=2 $\quad \omega_0 = \frac{2\pi * 2}{7} = 0.57\pi$

for k=3 $\quad \omega_0 = \frac{2\pi * 3}{7} = 0.85\pi$

for k=4 $\quad \omega_0 = \frac{2\pi * 4}{7} = 1.14\pi$

for k=5 $\quad \omega_0 = \frac{2\pi * 5}{7} = 1.4\pi$

for k=6 $\quad \omega_0 = \frac{2\pi * 6}{7} = 1.71\pi$



$$H(k) = \begin{cases} 0 & , for\ k = 0 \\ e^{-j3\frac{2\pi k}{7}} & , for\ k = 1,2 \\ 0 & , for\ k = 3,4 \\ e^{-j3\frac{2\pi k}{7}} & , for\ k = 5,6 \end{cases}$$

Design of linear phase FIR filter by frequency sampling technique

Step 3: Compute the N samples of impulse response h(n)

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k)e^{\frac{j2\pi nk}{N}}\right]\right], for\ N = odd$$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k)e^{\frac{j2\pi nk}{N}}\right]\right]$$

$$H(k) = \begin{cases} 0 & , for\ k = 0 \\ e^{-j3\frac{2\pi k}{7}} & , for\ k = 1,2 \\ 0 & , for\ k = 3,4 \\ e^{-j3\frac{2\pi k}{7}} & , for\ k = 5,6 \end{cases}$$

$$= \frac{1}{7}\left[0 + 2\sum_{k=1}^{3} Re\left[e^{-j3\frac{2\pi k}{7}}e^{\frac{j2\pi nk}{7}}\right]\right] = \frac{1}{7}\left[2\sum_{k=1}^{2} Re\left[e^{-j3\frac{2\pi k}{7}}e^{\frac{j2\pi nk}{7}}\right]\right]$$

$$= \frac{2}{7}\left[\sum_{k=1}^{2} Re\left[e^{j\frac{2\pi k}{7}(n-3)}\right]\right]$$

$$= \frac{2}{7}\left[2Re\left[e^{-j\frac{2\pi}{7}(n-3)}\right] + 2Re\left[e^{-j\frac{4\pi}{7}(n-3)}\right]\right]$$

$$h(n) = \frac{2}{7}\left[2\cos\left(\frac{2\pi}{7}(n-3)\right) + 2\cos\left(\frac{4\pi}{7}(n-3)\right)\right]$$

Design of linear phase FIR filter by frequency sampling technique

Now let's calculate h(n) for n = 0 to 6, using symmetric condition (h(n)=h(N-1-n))

for n=0

$$h(0) = \frac{2}{7}\left[2\cos\left(\frac{2\pi}{7}(0-3)\right) + 2\cos\left(\frac{4\pi}{7}(0-3)\right)\right] = -0.0792$$

for n=1

$$h(1) = \frac{2}{7}\left[2\cos\left(\frac{2\pi}{7}(1-3)\right) + 2\cos\left(\frac{4\pi}{7}(1-3)\right)\right] = -0.321$$

for n=2

$$h(2) = \frac{2}{7}\left[2\cos\left(\frac{2\pi}{7}(2-3)\right) + 2\cos\left(\frac{4\pi}{7}(2-3)\right)\right] = 0.1145$$

for n=3

$$h(3) = \frac{2}{7}\left[2\cos\left(\frac{2\pi}{7}(3-3)\right) + 2\cos\left(\frac{4\pi}{7}(3-3)\right)\right] = 0.571$$

for n=4

$$h(4) = h(7-1-4) = h(2) = 0.1145$$

for n=5

$$h(5) = h(7-1-5) = h(1) = -0.321$$

for n=6

$$h(6) = h(7-1-6) = h(0) = -0.0792$$

Design of linear phase FIR filter by frequency sampling technique

Step 4: Take Z-Transform of impulse response h(n) to get filter transfer function H(z)

$$H(Z) = \sum_{n=0}^{6} h(n)Z^{-n}$$

$$= h(0)Z^{0} + h(1)Z^{-1} + h(2)Z^{-2} + h(3)Z^{-3} + h(4)Z^{-4} + h(5)Z^{-5} + h(6)Z^{-6}$$

$$H(Z) = -0.0792(1 + Z^{-6}) - 0.321(Z^{-1} + Z^{-5}) + 0.1145(Z^{-2} + Z^{-4}) + 0.571Z^{-3}$$

$h(6) = h(7 - 1 - 6) = h(0) \quad = -0.0792$

$h(5) = h(7 - 1 - 5) = h(1) \quad = -0.321$

$h(4) = h(7 - 1 - 4) = h(2) \quad = 0.1145$

$h(3) \quad = 0.571$

Infinite Impulse Response (IIR) Filters

- *The FIR filters are non recursive type filters(present input depends on the present and previous inputs) where as IIR filters are recursive type (present input depends on the present, past and output samples)*
- *IIR (infinite impulse response) filters are generally chosen for applications where linear phase is not too important and memory is limited.*
- *They have been widely deployed in audio equalization, biomedical sensor signal processing, IoT/IIoT smart sensors and high-speed telecommunication/RF applications*
- *IIR filter have infinite-duration impulse responses, hence they can be matched to analog filters, all of which generally have infinitely long impulse responses.*
- *The basic techniques of IIR filter design transform well-known analog filters into digital filters using complex-valued mappings.*
- *First we design an antilog prototype filter and then transform the prototype to a digital filter, hence it is also called indirect method*

Infinite Impulse Response (IIR) Filters

- *An IIR filter is categorized by its theoretically infinite impulse response, Practically speaking, it is not possible to compute the output of an IIR using this equation. Therefore, the equation may be re-written in terms of a finite number of poles p and zeros q, as defined by the linear constant coefficient difference equation*

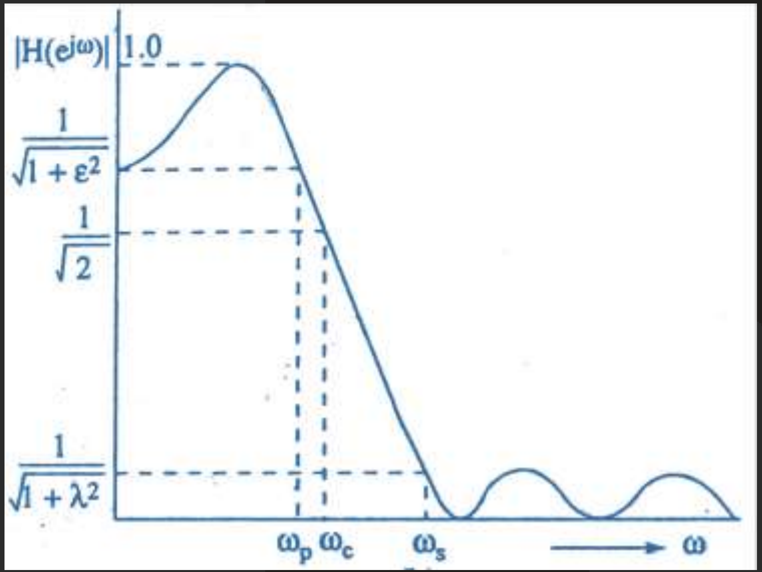
$$y(n) = \sum_{k=0}^{q} b_k x(n-k) - \sum_{l=1}^{p} a_k y(n-k)$$

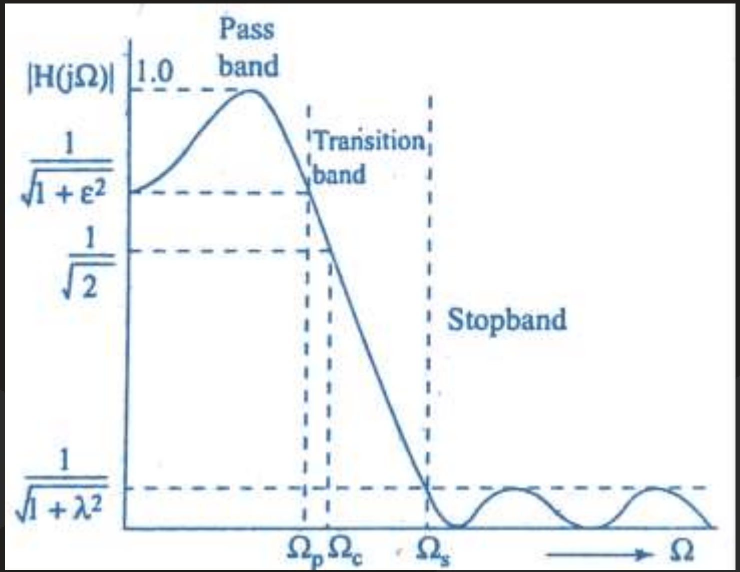
- *where, a(k) and b(k) are the filter's denominator and numerator polynomial coefficients, who's roots are equal to the filter's poles and zeros respectively. Thus, a relationship between the difference equation and the z-transform (transfer function) may therefore be defined by using the z-transform delay property such that,*

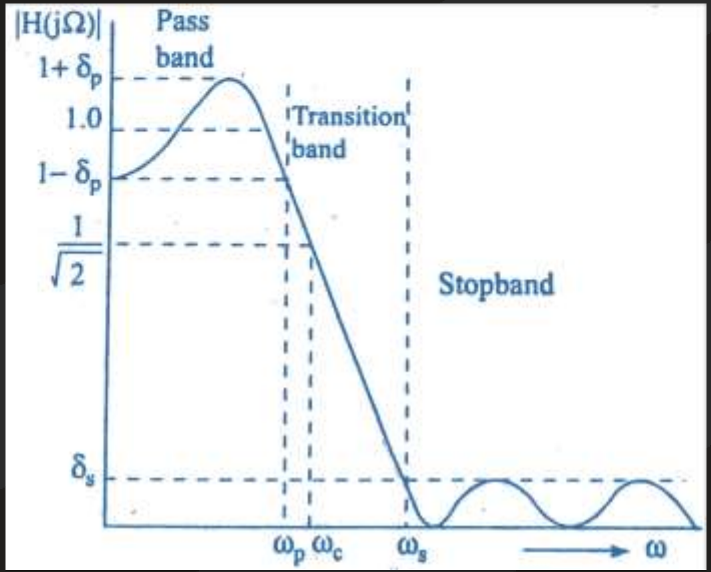
$$H(z) = \sum_{n=0}^{\infty} y(n) Z^{-n} = \frac{\sum_{k=0}^{q} b_k Z^{-k}}{1 + \sum_{k=1}^{p} a_k Z^{-k}}$$

- *As seen, the transfer function is a frequency domain representation of the filter.*
- *Notice also that the poles act on the output data, and the zeros on the input data.*
- *Since the poles act on the output data, and affect stability, it is essential that their radii remain inside the unit circle (i.e. <1) for BIBO (bounded input, bounded output) stability. The radii of the zeros are less critical, as they do not affect filter stability.*

Specifications for magnitude response of lowpass filter







Digital

ω_p → *Passband frequency (rad/samples)*

ω_s → *Stopband frequency (rad/samples)*

ω_c → *3dB cut off frequency (rad/samples)*

ε → *Passband parameter*

λ → *Stopband parameter*

Analog

Ω_p → *Passband frequency (rad/sec)*

Ω_s → *Stopband frequency (rad/sec)*

Ω_c → *3dB cut off frequency (rad/sec)*

δ_p → *Passband error tolerance*

δ_s → *max allowable magnitude in stop band*

Alternate specifications of lowpass filter

$$\varepsilon = \frac{2\sqrt{\delta_p}}{1-\delta_p}$$

$$\lambda = \frac{\sqrt{(1+\delta_p)^2 - \delta_s^2}}{\delta_s}$$

Design of steps of IIR Filters

1. *Map the desired digital filter specifications into those for an equivalent analog filter*
2. *Derive the analog transfer function for the analog prototype*
3. *Transform the transfer function of the analog prototype into an equivalent digital filter transfer function*

Analog lowpass filter design

Mainly there are two types of analog filter designs

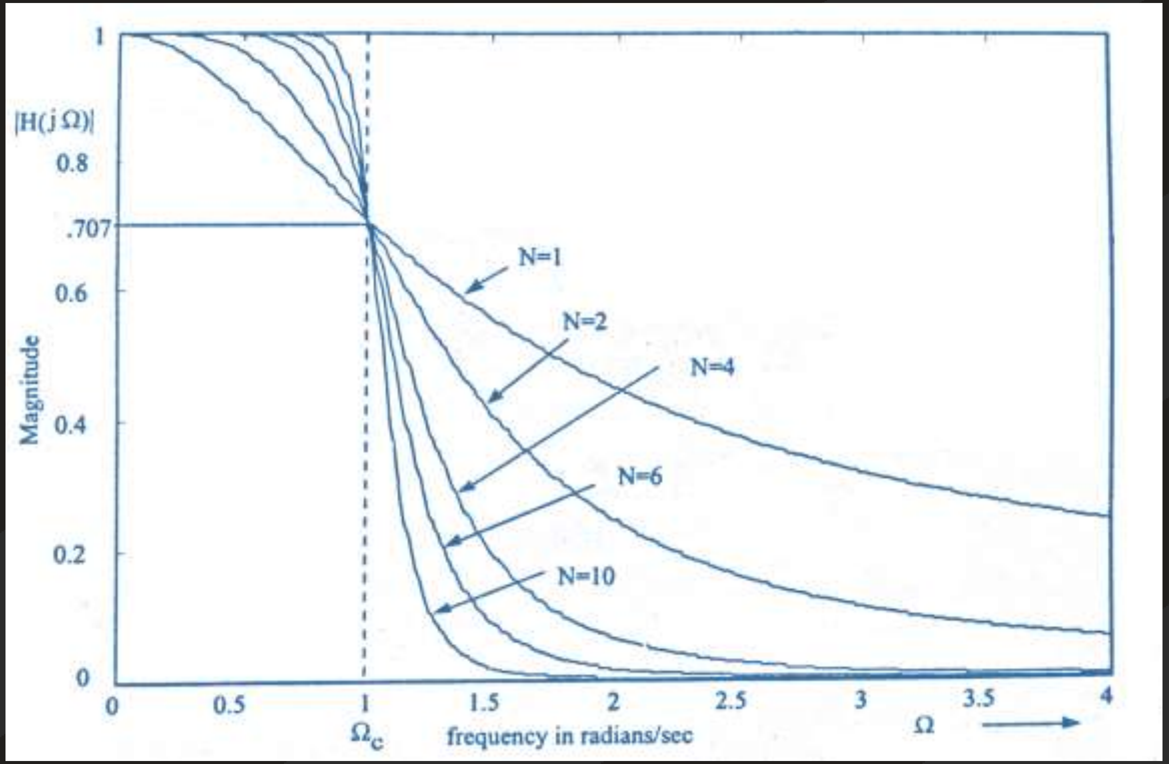
1. *Butterworth Filter*
2. *Chebyshev filter*

Analog low pass Butterworth Filter

The magnitude function of the lowpass Butterworth filter is

$$|H(j\Omega)| = \frac{1}{\left[1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]^{\frac{1}{2}}}$$

Where N is the order of the filter



Properties of Butterworth filters

1. *Butterworth filters are all pole design*
2. *The magnitude of normalised Butterworth filter is 1/√2 at cut off frequency* Ω_c
3. *The filter order specified the filter*
4. *Magnitude is maximally flat at the origin*
5. *As N increases the response approaches to ideal response*

Analog low pass Butterworth Filter

Order	Normalised transfer function
1	$\frac{1}{s+1}$
2	$\frac{1}{s^2+\sqrt{2}s+1}$
3	$\frac{1}{(s+1)(s^2+s+1)}$
4	$\frac{1}{(s^2+0.765s+1)(s^2+1.848s+1)}$
5	$\frac{1}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)}$
6	$\frac{1}{(s^2+1.931s+1)\left(s^2+\sqrt{2s}+1\right)(s^2+0.517s+1)}$

Order of the Butterworth filter

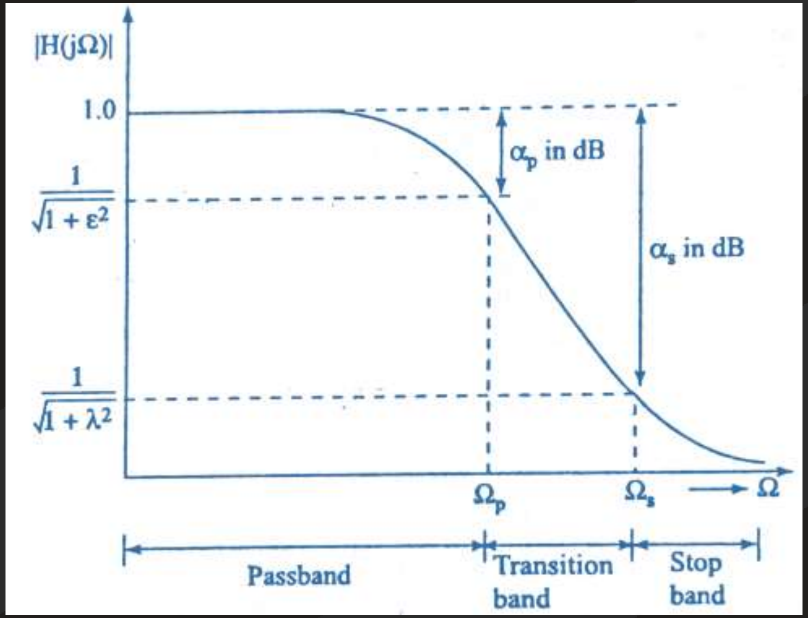
Let the maximum passband attenuation in positive dB is α_p (<3dB) at passband frequency Ω_p and α_s is the minimum stopband attenuation at stopband frequency Ω_s . The magnitude function can be written as

$$|H(j\Omega)| = \frac{1}{\left[1 + \varepsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N}\right]^{\frac{1}{2}}}$$

$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

Taking log on both sides

$$20 \log H(j\Omega) = 10 \log 1 - 10 \log(1 + \varepsilon^2)$$



$\Omega_p \rightarrow$ *Passband frequency (rad/sec)*

$\Omega_s \rightarrow$ *Stopband frequency (rad/sec)*

$\varepsilon \rightarrow$ *Passband parameter*

$\lambda \rightarrow$ *Stopband parameter*

$\alpha_p \rightarrow$ *Passband attenuation*

$\alpha_s \rightarrow$ *Stopband attenuation*

Order of the Butterworth filter

At $\Omega = \Omega_p$, $20 \log H(j\Omega) = -\alpha_p$

$$\alpha_p = 10 \log(1 + \varepsilon^2)$$

$$0.1\alpha_p = \log(1 + \varepsilon^2)$$

Taking antilog on both sides

$$10^{0.1\alpha_p} = 1 + \varepsilon^2$$

$$\varepsilon^2 = 10^{0.1\alpha_p} - 1$$

$$\varepsilon = \left(10^{0.1\alpha_p} - 1\right)^{\frac{1}{2}}$$

At $\Omega = \Omega_s$, $20 \log H(j\Omega) = -\alpha_s$

$$\alpha_s = 10 \log\left(1 + \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p}\right)^{2N}\right)$$

Taking antilog on both sides

$$10^{0.1\alpha_s} - 1 = \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p}\right)^{2N}$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s} - 1}{\varepsilon^2}$$

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}$$

Taking log on both sides and finding the value of N

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

or

$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

Steps to design an analog Butterworth filter

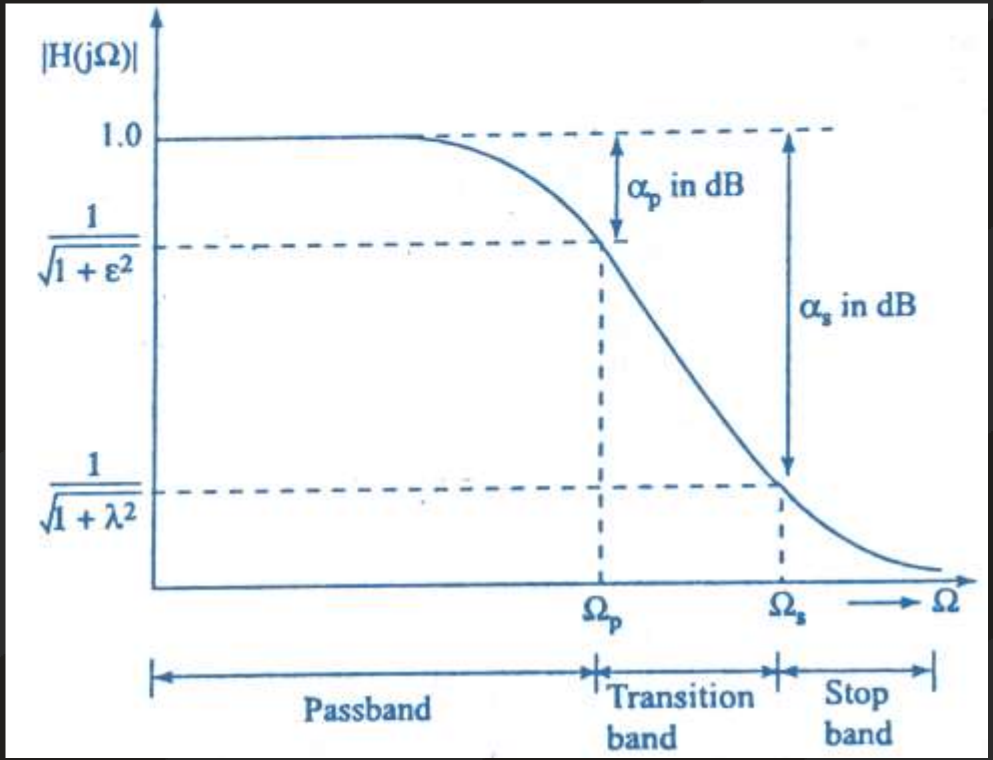
1. *Find the order of the filter N &round off to higher integer*
2. *Find the transfer function H(s) for* Ω_c*=1 rad/sec for the values of N*
3. *Calculate value of cut-off frequency* Ω_c
4. *Find transfer function Ha(s) for the value of* Ω_c *calculated by substituting* $s = \frac{s}{\Omega_c}$ *in H(s)*

Design an analog Butterworth filter

Q) For given specifications design an analog Butterworth filter

$$0.9 \leq |H(j\Omega)| \leq 1 \; for \; 0 \leq \Omega \leq 0.2\pi$$

$$|H(j\Omega)| \leq 0.2 \; for \; 0.4\pi \leq \Omega \leq \pi$$



Solution

$\Omega_p = 0.2\pi$ $\quad \frac{1}{\sqrt{1+\varepsilon^2}} = 0.9$ $\quad \varepsilon = 0.484$

$\Omega_s = 0.4\pi$ $\quad \frac{1}{\sqrt{1+\lambda^2}} = 0.2$ $\quad \lambda = 4.898$

Step 1: Find the order of the filter N & round off to higher integer

$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \geq \frac{\log\left(\frac{4.898}{0.484}\right)}{\log\left(\frac{0.4\pi}{0.2\pi}\right)} = 3.34 \qquad N \approx 4$$

Design an analog Butterworth filter

Step 2: Find the transfer function H(s) for Ωc=1 rad/sec for the values of N

$$H(S_n) = \frac{1}{(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)}$$

Order	Normalised transfer function
1	$\frac{1}{s+1}$
2	$\frac{1}{s^2+\sqrt{2}s+1}$
3	$\frac{1}{(s+1)(s^2+s+1)}$
4	$\frac{1}{(s^2+0.765s+1)(s^2+1.848s+1)}$
5	$\frac{1}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)}$
6	$\frac{1}{(s^2+1.931s+1)(s^2+\sqrt{2}s+1)(s^2+0.517s+1)}$

Step 3: Calculate value of cut-off frequency Ωc

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p} - 1\right)^{\frac{1}{2N}}} = \frac{0.2\pi}{\varepsilon^{\frac{1}{4}}} = 0.24\pi$$

$$\varepsilon = \left(10^{0.1\alpha_p} - 1\right)^{\frac{1}{2}}$$

Design an analog Butterworth filter

Step 4: Find transfer function H(s) for the value of Ω_c calculated by substituting $s = \frac{s}{\Omega_c}$ in H(s)

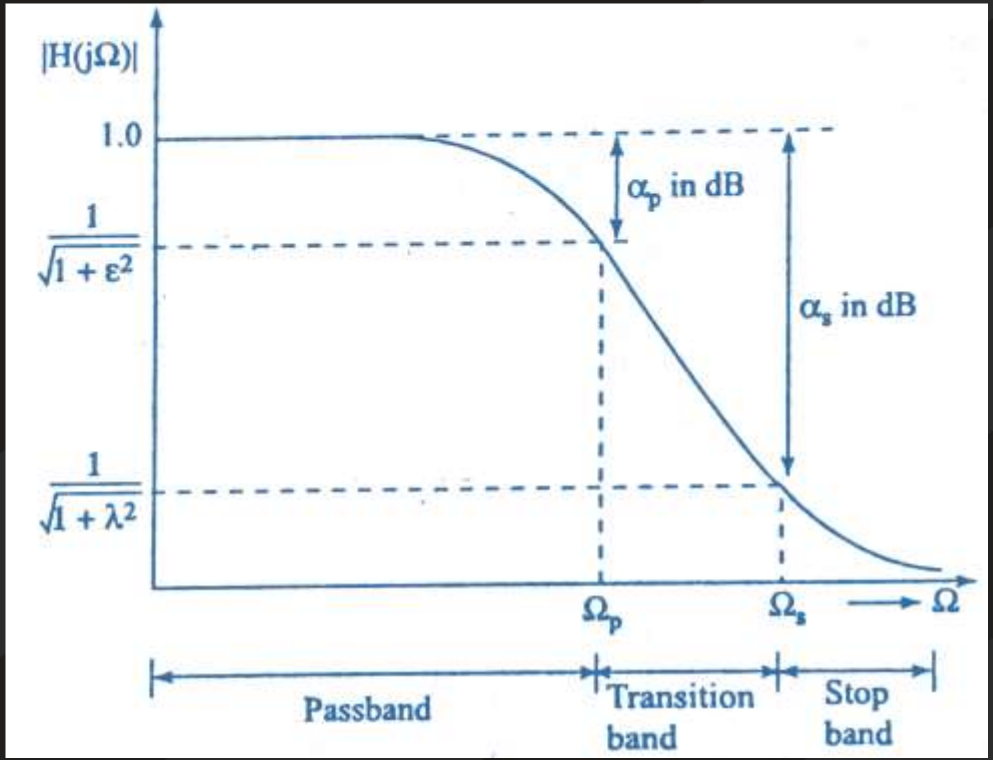
$$H(S_n) = \frac{1}{(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)}$$

$$H(s) = \frac{1}{\left(\left(\frac{s}{0.24\pi}\right)^2 + 0.765\frac{s}{0.24\pi} + 1\right)\left(\left(\frac{s}{0.24\pi}\right)^2 + 1.848\frac{s}{0.24\pi} + 1\right)}$$

$$H(s) = \frac{0.323}{(s^2 + 0.577s + 0.057\pi^2)(s^2 + 1.39s + 0.0576\pi^2)}$$

Design an analog Butterworth filter

Q) Design an analog Butterworth filter that has a -2dB passband attenuation at a frequency of 20 rad/sec and at least -10dB stopband attenuation at 30 rad/sec



Solution

$$\Omega_p = 20\, rad/sec \qquad |\alpha_p| = 2dB$$

$$\Omega_s = 30\, rad/sec \qquad |\alpha_s| = 10dB$$

Step 1: Find the order of the filter N & round off to higher integer

$$N \geq \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \geq \frac{\log\sqrt{\frac{10^{0.1*10}-1}{10^{0.1*2}-1}}}{\log\left(\frac{30}{20}\right)} = 3.37 \qquad N \approx 4$$

Design an analog Butterworth filter

Step 2: Find the transfer function H(s) for Ωc=1 rad/sec for the values of N

$$H(S_n) = \frac{1}{(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)}$$

Order	Normalised transfer function
1	$\frac{1}{s+1}$
2	$\frac{1}{s^2+\sqrt{2}s+1}$
3	$\frac{1}{(s+1)(s^2+s+1)}$
4	$\frac{1}{(s^2+0.765s+1)(s^2+1.848s+1)}$
5	$\frac{1}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)}$
6	$\frac{1}{(s^2+1.931s+1)(s^2+\sqrt{2}s+1)(s^2+0.517s+1)}$

Step 3: Calculate value of cut-off frequency Ωc

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{\frac{1}{2N}}} = \frac{20}{(10^{0.1*2} - 1)^{\frac{1}{2*4}}} = 21.386$$

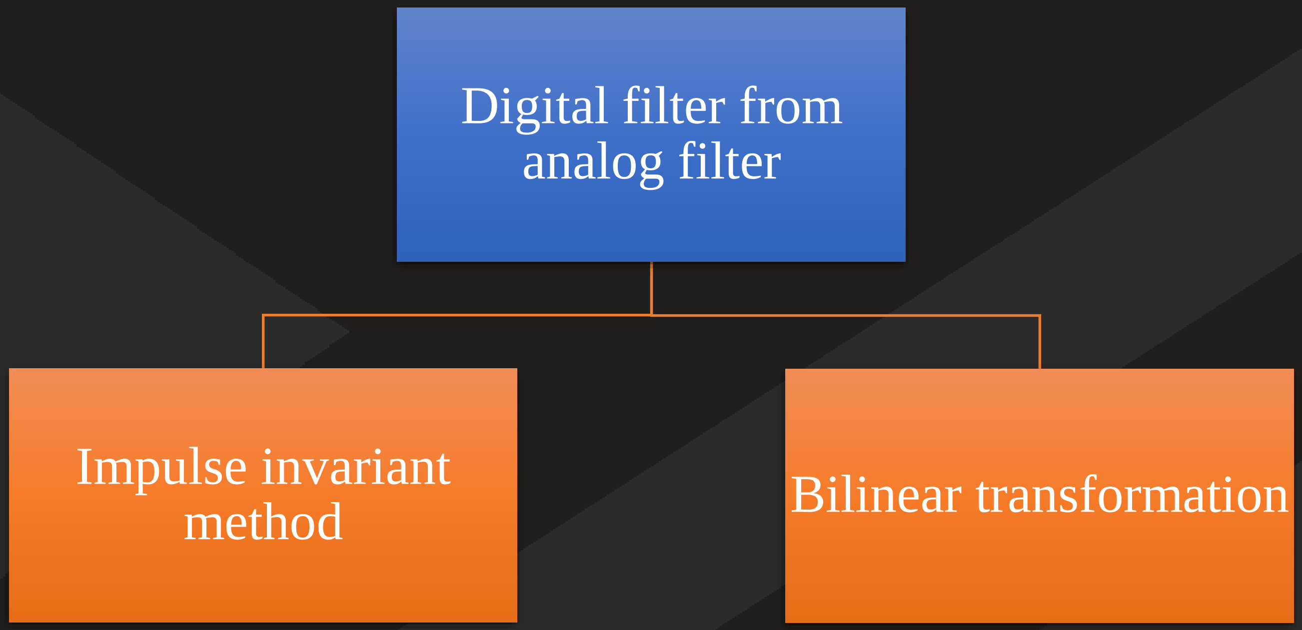
Design an analog Butterworth filter

Step 4: Find transfer function Ha(s) for the value of Ω_c calculated by substituting $s = \frac{s}{\Omega_c}$ in H(s)

$$H(S_n) = \frac{1}{\left(\left(\frac{s}{21.386}\right)^2 + 0.765\frac{s}{21.386} + 1\right)\left(\left(\frac{s}{21.386}\right)^2 + 1.848\frac{s}{21.386} + 1\right)}$$

$$H(S_n) = \frac{0.20921 \times 10^6}{(s^2 + 16.368s + 457.39)(s^2 + 39.51s + 457.394)}$$

Design digital filter from analog filter



Design of IIR filter using Impulse invariant method

- *Here we require that the impulse response of the discrete system (digital filter) be the discrete version of the impulse response of the analogue system (filter)*
- *Hence the name impulse invariant*
- *In impulse invariant method the IIR filter is designed such that the unit impulse response h(n) of digital filter is the sampled version of impulse response of analog filter*

Z transform

$$H(Z) = \sum_{n=0}^{N} h(n)Z^{-n}$$

For impulse invariant method we do the mapping as

$$H(Z)|_{Z=e^{sT}} = \sum_{n=0}^{N} h(n)e^{-sTn}$$

$$S = \sigma + j\Omega \qquad Z = re^{j\omega}$$

$$re^{j\omega} = e^{(\sigma + j\Omega)T}$$

Equating real and imaginary parts

$$r = e^{\sigma T} \qquad \omega = \Omega T$$

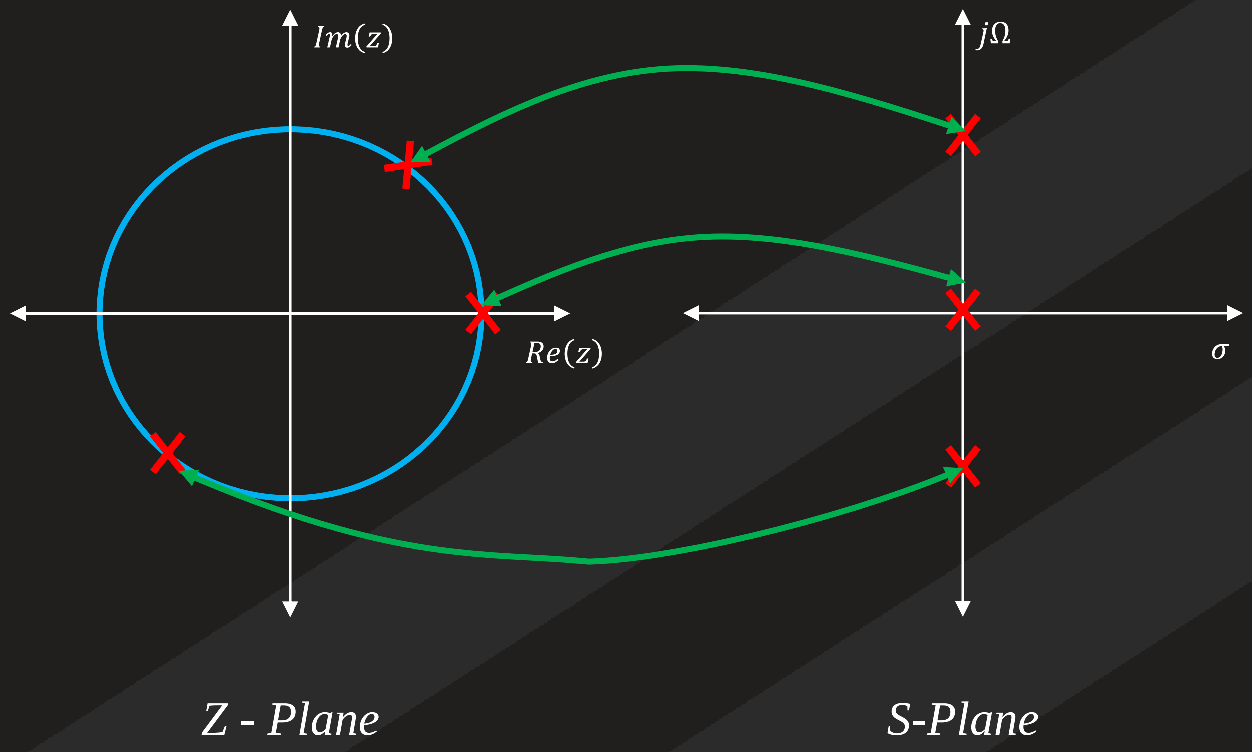
Real part of analog pole =radius of Z-plane pole

Imaginary part of analog pole = angle of digital pole

Case 1 :

$$\sigma = 0$$

$$r = e^{0T} = 1$$

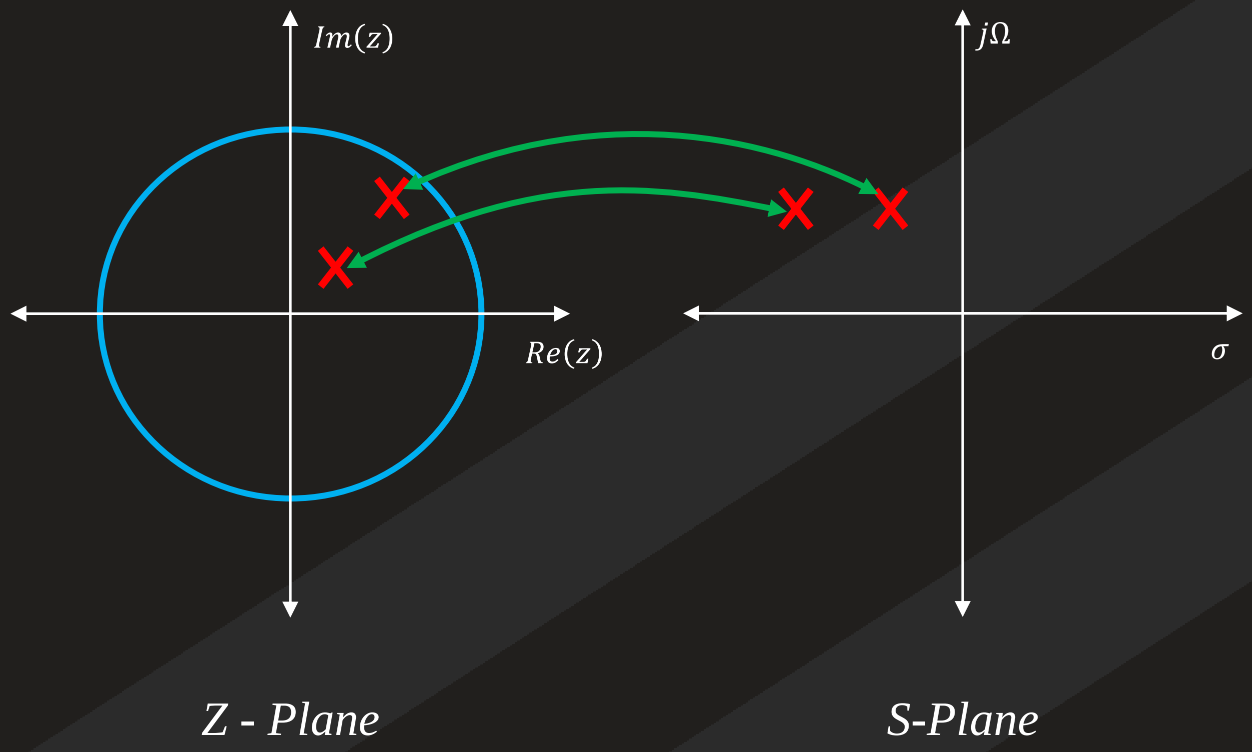


Impulse invariant mapping map poles from s-plane's jΩ axis to Z-plane's unit circle

Case 2 :

$$\sigma < 0\ (poles\ in\ left\ half\ of\ S - plane)$$

$$r = e^{\sigma T} < 1$$

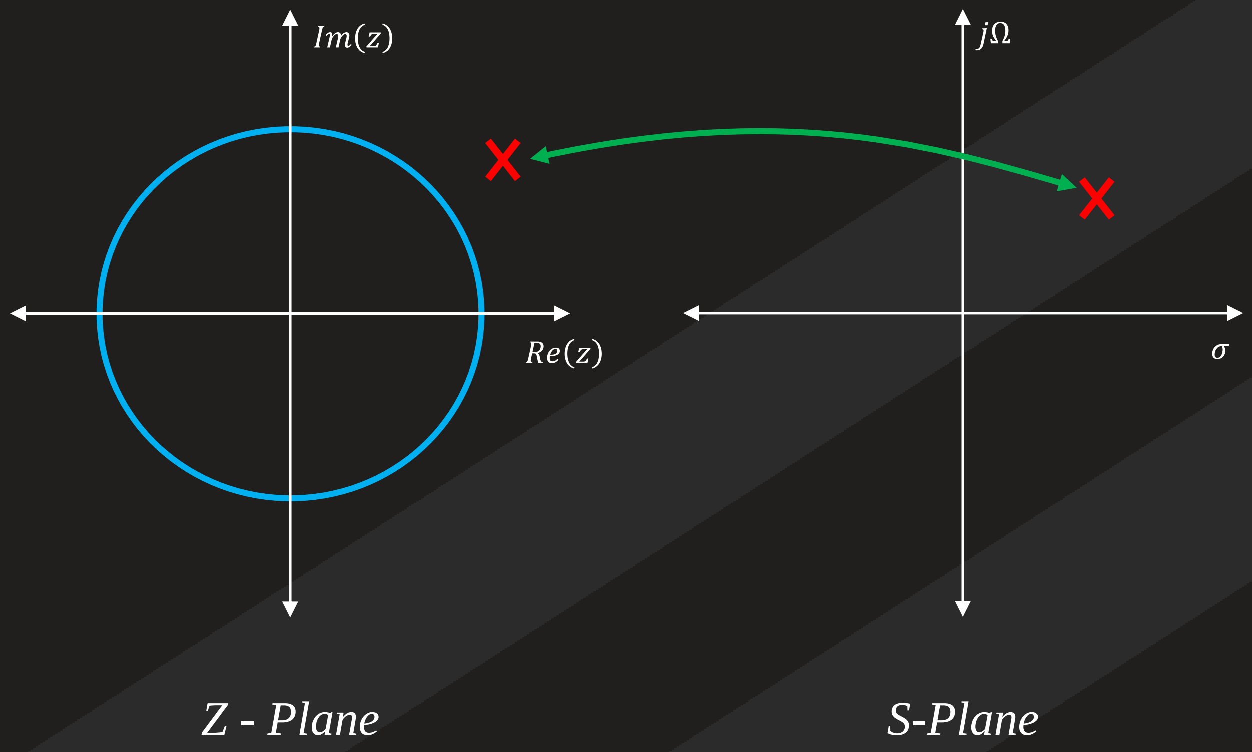


All S-plane poles with –ve real parts map to Z–plane poles inside unit circle

Case 3 :

$\sigma > 0$ (poles in right half of $S-plane$)

$r = e^{\sigma T} > 1$



Poles in right half of S-plane map to digital poles outside unit circle

Let $H_a(s)$ is the system function of analog filter

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{S - P_k}$$

$P_k \rightarrow Poles\ of\ analog\ filter$

$C_k \rightarrow Coefficients\ in\ partial fraction\ expansion$

$$L\left[\frac{1}{s-a}\right] = e^{at}$$

Taking inverse Laplace transform

$$h_a(t) = \sum_{k=1}^{N} C_k e^{P_k t}$$

Sample $h_a(t)$ at $t=nT$

$$h(n) = h(nT) = \sum_{k=1}^{N} C_k e^{P_k nT}$$

Now taking Z - Transform

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$H(z) = \sum_{n=0}^{\infty} \sum_{k=1}^{N} C_k e^{P_k nT} z^{-n}$$

$$= \sum_{k=1}^{N} C_k \sum_{n=0}^{\infty} (e^{P_k T} z^{-1})^n$$

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{S - P_k}$$

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

Steps to design IIR filter using Impulse invariant method

1. *For the given specification find Ha(s), the transfer function of analog filter*
2. *Select the sampling rate of the digital filter, T seconds/sample*
3. *Express the analog filter transfer function as the sum of single pole filters*

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{S - P_k}$$

4. *Compute the Z transform of the digital filter by using the formula*

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

$$H(z) = \sum_{k=1}^{N} \frac{T C_k}{1 - e^{P_k T} z^{-1}} \quad \text{for } T < 1$$

Design of IIR filter by impulse invariant technique

Q) For the analog transfer function $H(s) = \frac{2}{(s+1)(s+2)}$ *Determine H(z) using impulse invariance method. Assume T=1sec*

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{S - P_k}$$

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

Solution

$$H(s) = \frac{2}{(s+1)(s+2)}$$

Using partial fraction method

$$H(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$

$$2 = A(s+2) + B(s+1)$$

At s = -1

$$A = 2$$

At s = -2

$$B = -2$$

$$H(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)}$$

$$H(s) = \frac{2}{(s-(-1))} - \frac{2}{(s-(-2))}$$

For T=1 sec

$$\frac{2}{(s-(-1))} = \frac{2}{1 - e^{-1} z^{-1}}$$

$$\frac{2}{(s-(-2))} = \frac{2}{1 - e^{-2} z^{-1}}$$

$$H(z) = \frac{2}{1 - e^{-1} z^{-1}} - \frac{2}{1 - e^{-2} z^{-1}}$$

$$H(z) = \frac{0.465 z^{-1}}{1 - 0.503 z^{-1} + 0.0497 z^{-2}}$$

Design of IIR filter by impulse invariant technique

Q) An anlog filter has a transfer function $H(s) = \frac{10}{s^2+7s+10}$ Design a digital filter equivalent to this using impulse invariant method for T=0.2 sec

Solution

$$H(s) = \frac{10}{s^2 + 7s + 10}$$

Using partial fraction method

$$\frac{10}{s^2 + 7s + 10} = \frac{A}{(s+5)} + \frac{B}{(s+2)}$$

$$10 = A(s+2) + B(s+5)$$

At s = -5 At s = -2

$$A = -3.33 \qquad B = 3.33$$

$$H(s) = \frac{-3.33}{(s+5)} + \frac{3.33}{(s+2)}$$

$$H(s) = \frac{-3.33}{(s-(-5))} + \frac{3.33}{(s-(-2))}$$

$$H(s) = \sum_{k=1}^{N} \frac{C_k}{S - P_k}$$

$$H(z) = \sum_{k=1}^{N} \frac{C_k}{1 - e^{P_k T} z^{-1}}$$

For T=0.2 sec

$$\frac{-3.33}{(s-(-5))} = \frac{-3.33}{1 - e^{-5*0.2} z^{-1}}$$

$$\frac{3.33}{(s-(-2))} = \frac{3.33}{1 - e^{-2*02} z^{-1}}$$

$$H(z) = \left[\frac{-3.33}{1 - e^{-1} z^{-1}} - \frac{3.33}{1 - e^{-0.4} z^{-1}}\right] 0.2$$

$$H(z) = \frac{0.2012 z^{-1}}{1 - 1.0378 z^{-1} + 0.247 z^{-2}}$$

Design of IIR filter by impulse invariant technique

Q) Apply impulse invariant method and find H(z) for $H(s) = \frac{s+a}{(s+a)^2+b^2}$

$$e^{at}\cos bt \overset{L}{\Leftrightarrow} \frac{(s-a)}{(s-a)^2+b^2}$$

Solution

$$H(s) = \frac{s+a}{(s+a)^2+b^2}$$

Inverse Laplace of the given function

$$h(t) = e^{-at}\cos(bt)$$

For sampling the function substitute t=nT

$$h(nT) = e^{-anT}\cos(bnT)$$

Taking Z-transform

$$H(z) = \sum_{n=0}^{\infty} e^{-anT}\cos(bnT)\, z^{-n}$$

$$H(z) = \sum_{n=0}^{\infty} e^{-anT} z^{-n}\left(\frac{e^{jbnT}+e^{-jbnT}}{2}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-aT}e^{jbT}z^{-1}\right)^n + \left(e^{-aT}e^{-jbT}z^{-1}\right)^n$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-(a-jb)T}z^{-1}\right)^n + \left(e^{-(a+jb)T}z^{-1}\right)^n$$

$$\sum_{n=0}^{\infty} a^n \Leftrightarrow \frac{1}{1-a}$$

$$H(z) = \frac{1}{2}\left[\frac{1}{1-e^{-(a-jb)T}z^{-1}} + \frac{1}{1-e^{-(a+jb)T}z^{-1}}\right]$$

$$H(z) = \frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)\,z^{-1}+e^{-2aT}z^{-2}}$$

Steps to design IIR filter using Bilinear transformation method

The basis operation is to convert an analogue filter H(s) into an equivalent digital filter H(z) by using bilinear approximation

1. *From the given specifications find pre-warping analog frequency using formula*

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

2. *Using the analog frequency find H(s) of the analog filter*
3. *Select the sampling rate of the digital filter, call it T seconds per sample*
4. *Substitute* $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ *into the transfer function found in step 2*

Design of IIR filter by impulse invariant technique

Q) Apply bilinear transformation to $H(s) = \frac{2}{(s+1)+(s+2)}$

Solution

$$H(s) = \frac{2}{(s+1)+(s+2)}$$

Substitute $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ *at T=1sec*

$$H(z) = \frac{2}{\left(2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+1\right)+\left(2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+2\right)}$$

$$= \frac{2}{\left(2\left[\frac{[1-z^{-1}]+[1+z^{-1}]}{1+z^{-1}}\right]\right)+\left(2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]+2\right)}$$

$$= \frac{2(1+z^{-1})^2}{(2-2z^{-1}+1+z^{-1})+(2-2z^{-1}+2+2z^{-1})}$$

$$= \frac{(1+z^{-1})^2}{(3-z^{-1})2}$$

$$H(z) = \frac{(1+z^{-1})}{(1-0.33z^{-1})}$$

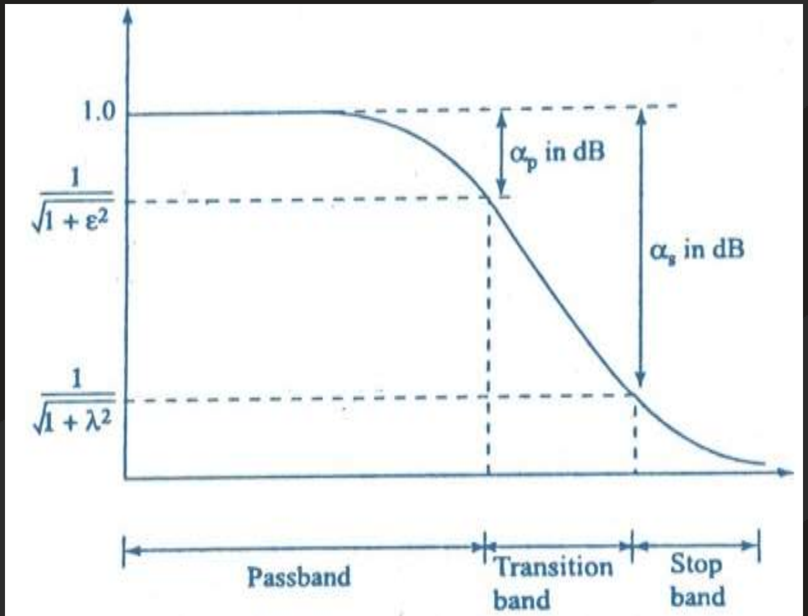
Design an analog Butterworth filter

Q) Design a digital analog Butterworth filter satisfying the constraints

$$0.707 \leq |H(j\omega)| \leq 1 \; for \; 0 \leq \omega \leq \frac{\pi}{2}$$

$$|H(j\omega)| \leq 0.2 \; for \frac{3\pi}{4} \leq \omega \leq \pi$$

using bilinear transformation. Take T=1sec



Solution

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.707 \qquad \varepsilon = 1 \qquad \omega_p = \frac{\pi}{2}$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2 \qquad \lambda = 4.89 \qquad \omega_s = \frac{3\pi}{4}$$

Step 1: Step 1: From the given specifications find pre-warping analog frequency using formula $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} \qquad \Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2}$$

$$\Omega_s = \frac{2}{1}\tan\frac{\frac{3\pi}{4}}{2} \qquad \Omega_p = \frac{2}{1}\tan\frac{\frac{\pi}{2}}{2}$$

$$\Omega_s = 2\tan\frac{3\pi}{8} \qquad \Omega_p = 2\tan\frac{\pi}{4}$$

$$\frac{\Omega_s}{\Omega_p} = \frac{2\tan\frac{3\pi}{8}}{2\tan\frac{\pi}{4}} = 2.414$$

Design an analog Butterworth filter

Step 2: Using the analog frequency find H(s) of the analog filter

$$N \geq \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \quad \geq \frac{\log\left(\frac{4.89}{1}\right)}{\log(2.414)} \quad \geq 1.80$$

$$N = 2$$

$$H_a(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{\frac{1}{N}}} \quad = \frac{2\tan\frac{\pi}{4}}{1^{\frac{1}{2}}} \quad = 2\, rad/sec$$

Order	Normalised transfer function
1	$\frac{1}{s+1}$
2	$\frac{1}{s^2+\sqrt{2}s+1}$
3	$\frac{1}{(s+1)(s^2+s+1)}$
4	$\frac{1}{(s^2+0.765s+1)(s^2+1.848s+1)}$
5	$\frac{1}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)}$
6	$\frac{1}{(s^2+1.931s+1)(s^2+\sqrt{2}s+1)(s^2+0.517s+1)}$

To find H(s) substitute $s = \frac{s}{\Omega_c}$

$$H(s) = \frac{1}{\frac{s^2}{4} + \sqrt{2}\frac{s}{2} + 1}$$

$$H(s) = \frac{4}{s^2 + 2.828s + 4}$$

Design an analog Butterworth filter

Step 4: Substitute $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ *into the transfer function*

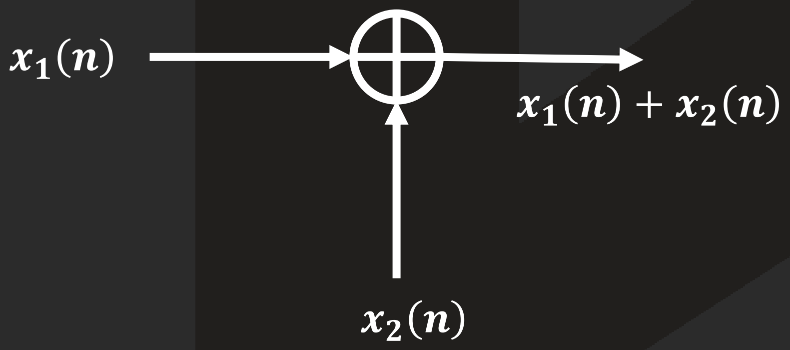
$$H(s) = \frac{1}{s^2 + 2.828s + 4}$$

$$= \frac{1}{\left[2\frac{1-z^{-1}}{1+z^{-1}}\right]^2 + 2.828\left(2\frac{1-z^{-1}}{1+z^{-1}}\right) + 4}$$

$$H(s) = \frac{4[1+z^{-1}]}{4[1-z^{-1}] + 5.656[1-z^{-2}] + 4[1+z^{-1}]}$$

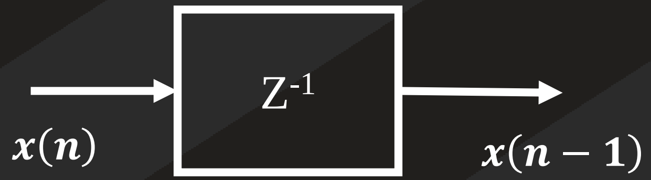
MODULE 4 Realisation of discrete time system

- A digital filter transfer function can be realized in a variety ways
  - Realization of FIR filters
  - Realization of IIR filters
- Basic elements required for implementation of an LTI digital system are adder , multiplier and memory for storing elements
- In digital implementation the delay operation can be implemented by providing a storage register by each unit delay is required



Adder

Multiplier



Unit delay

FIR filter realization

- Mainly four types of realisation are there
  1. Direct form realisation
  2. Cascade form realisation
  3. Linear phase realisation
  4. Lattice structure realisation

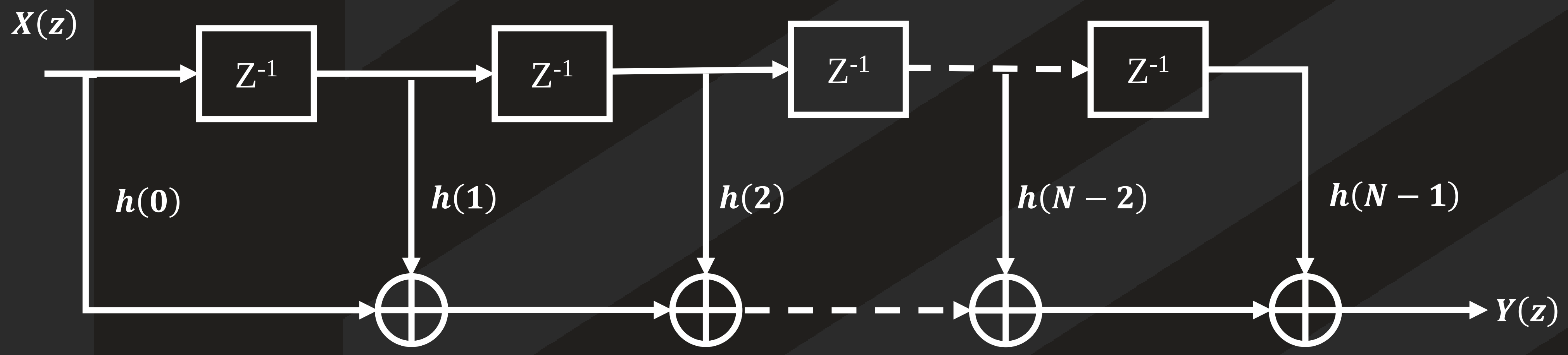
Direct form realization

The direct form of FIR may be obtained by using the equation of linear convolution

$$y(k) = \sum_{k=0}^{N-1} h(k)x(n-k) \qquad = h(0)x(n) + h(1)x(n-1) + \cdots + h(N-1)x(n-N-1)$$

Taking Z-transform

$$Y(z) = h(0)X(z) + h(1)X(z)z^{-1} + \cdots + h(N-1)X(z)z^{-(N-1)}$$

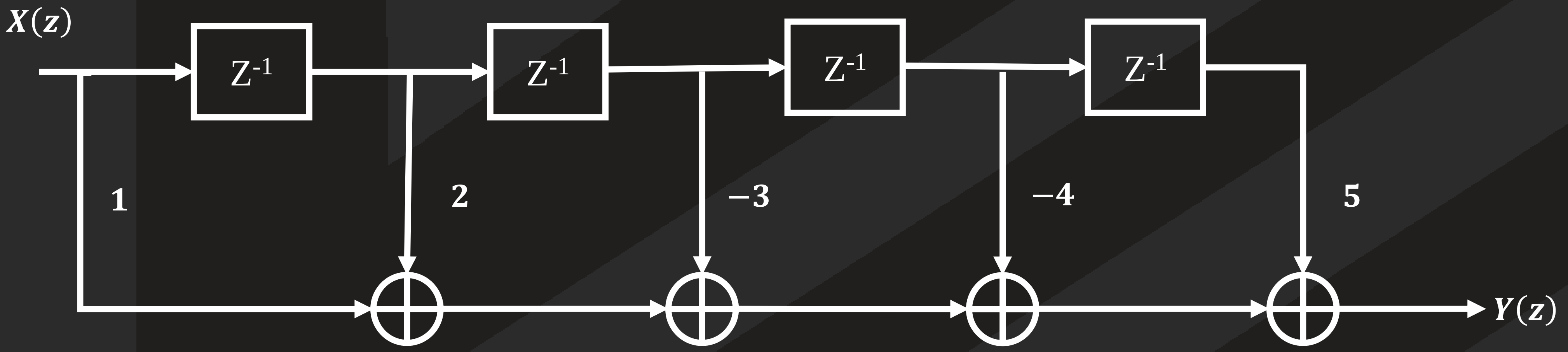


Q) *Determine the direct form realization of system function*
$H(z) = 1 + 2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}$

$$H(z) = \frac{Y(z)}{X(z)}$$

$$H(z) = 1 + 2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}$$

$$Y(z) = X(z)[1 + 2z^{-1} - 3z^{-2} - 4z^{-3} + 5z^{-4}]$$



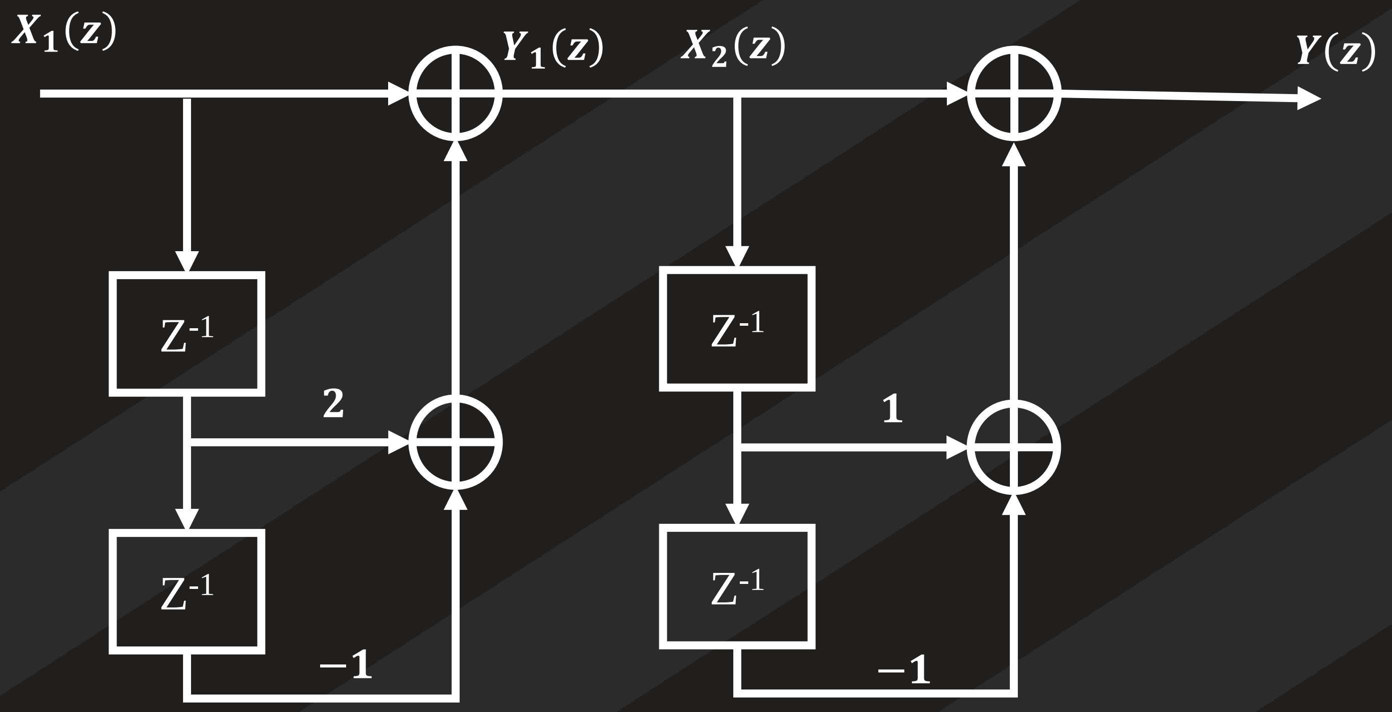
Cascade form realization

Q) Obtain cascade form realization of system function $H(z) = (1 + 2z^{-1} - z^{-2})(1 + z^{-1} - z^{-2})$

$$H(z) = H_1(z)H_2(z)$$

$$H_1(z) = \frac{Y_1(z)}{X_1(z)}$$

$$H_2(z) = \frac{Y_2(z)}{X_2(z)}$$



$$Y_1(z) = X_1(z)[1 + 2z^{-1} - z^{-2}]$$

$$Y_1(z) = X_1(z) + 2X(z)z^{-1} - X(z)z^{-2}$$

$$Y_2(z) = X_2(z)[1 + z^{-1} - z^{-2}]$$

$$Y_2(z) = X_2(z) + X_2(z)z^{-1} - X_2(z)z^{-2}$$

Cascade form realization

Q) Obtain cascade form realization of system function $H(z) = 1 + \frac{5}{2}z^{-1} + 2z^{-2} + 2z^{-3}$

$$H(z) = 1 + \frac{5}{2}z^{-1} + 2z^{-2} + 2z^{-3}$$

$$= \frac{z^3}{z^3}\left[1 + \frac{5}{2}z^{-1} + 2z^{-2} + 2z^{-3}\right]$$

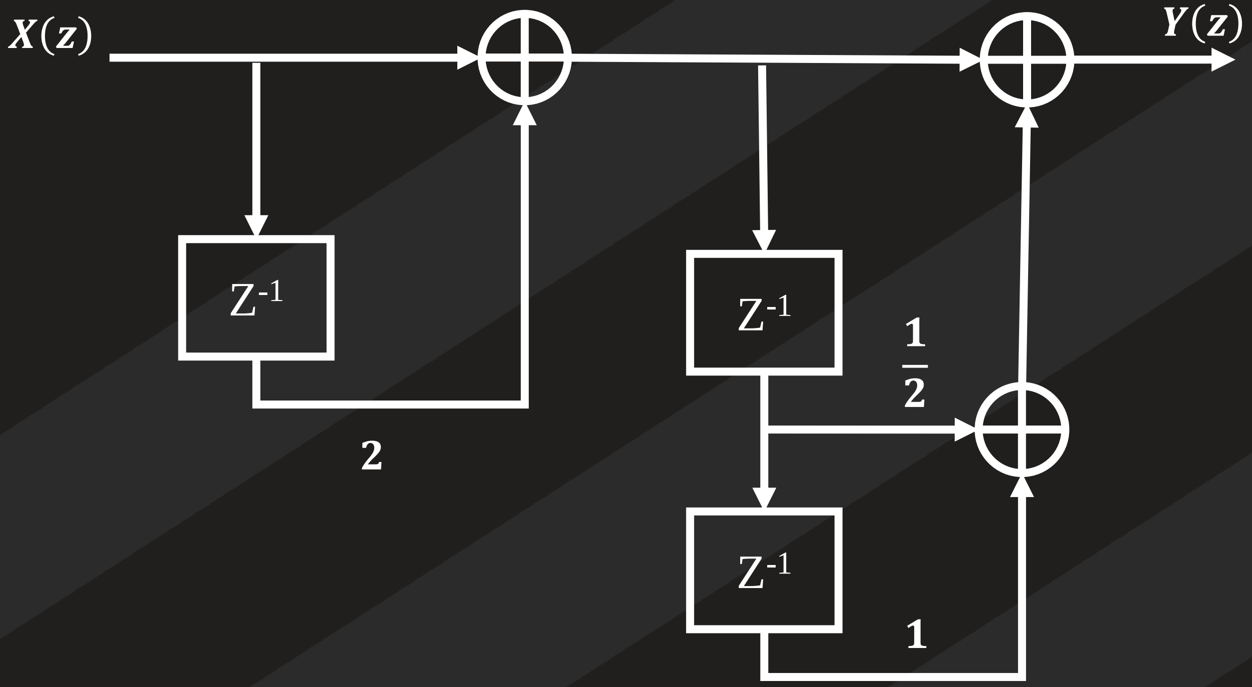
$$= \frac{1}{z^3}\left[z^3 + \frac{5}{2}z^2 + 2z^1 + 2\right]$$

The term inside the bracket equal to zero when z=-2

So the first term can be (z+2)

$$= \frac{1}{z^3}\left[(z+2)\left(z^2 + \frac{1}{2}z + 1\right)\right]$$

$$= \frac{1}{z^3}\left[z(1 + 2z^{-1})z^2\left(1 + \frac{1}{2}z^{-1} + z^{-2}\right)\right]$$



Linear phase realization

Q) Realise the system function $H(z) = \frac{1}{2} + \frac{1}{3}z^{-1} + z^{-2} + \frac{1}{4}z^{-3} + z^{-4} + \frac{1}{3}z^{-5} + \frac{1}{2}z^{-6}$

$h(0)$ $h(1)$ $h(2)$ $h(3)$ $h(4)$ $h(5)$ $h(6)$

For a linear phase FIR filter $h(n) = h(N-1-n)$

$N = 7$

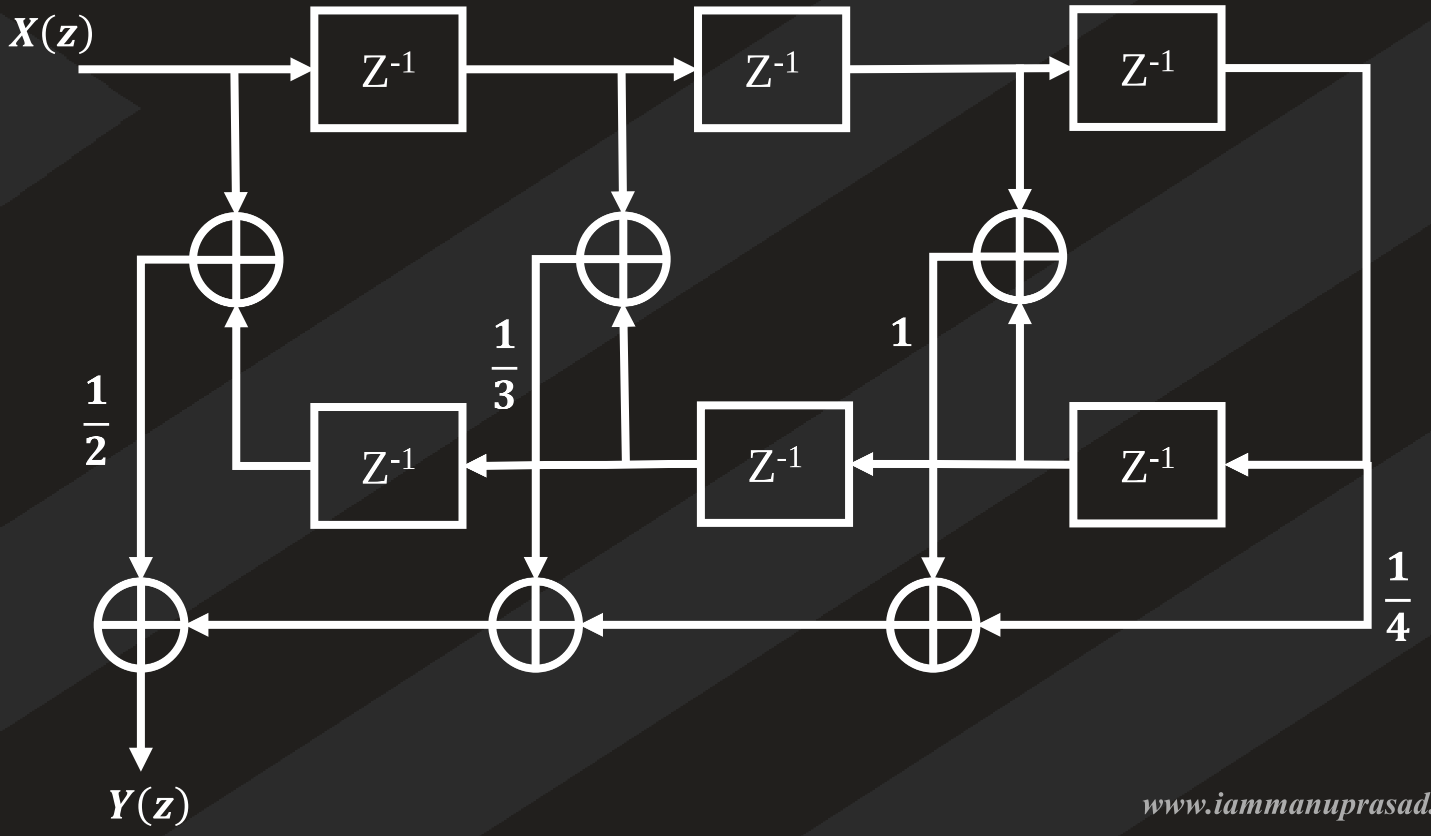
$h(0) = h(7-1-0) = h(6) = \frac{1}{2}$

$h(1) = h(7-1-1) = h(5) = \frac{1}{3}$

$h(2) = h(7-1-2) = h(4) = 1$

$h(3) = h(7-1-3) = h(3) = \frac{1}{4}$

$$H(z) = \frac{1}{2}(1 + z^{-6}) + \frac{1}{3}(z^{-1} + z^{-5}) + 1.(z^{-2} + z^{-4}) + \frac{1}{4}(z^{-3})$$



Linear phase realization

Q) *Realise the system function using minimum number of multiplier*

$$H(z) = 1 + \frac{1}{3}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{3}z^{-4} + z^{-5}$$

$h(0)$ $h(1)$ $h(2)$ $h(3)$ $h(4)$ $h(5)$

For a linear phase FIR filter $h(n) = h(N-1-n)$

$N = 6$

$h(0) = h(6-1-0) = h(5) = 1$

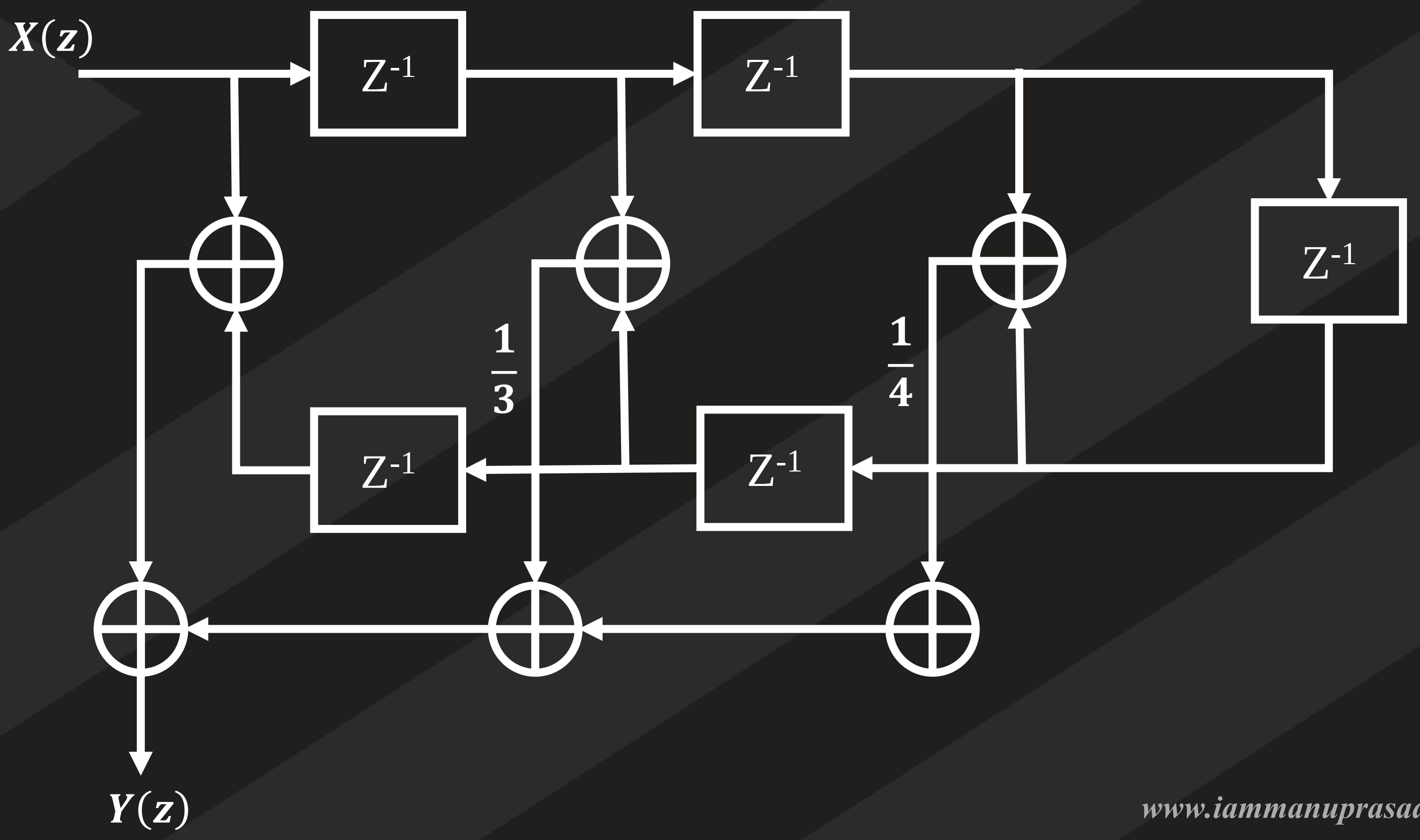
$h(1) = h(6-1-1) = h(4) = \frac{1}{3}$

$h(2) = h(6-1-2) = h(3) = \frac{1}{4}$

$H(z)$

$= 1(1+z^{-5}) + \frac{1}{3}(z^{-1}+z^{-4})$

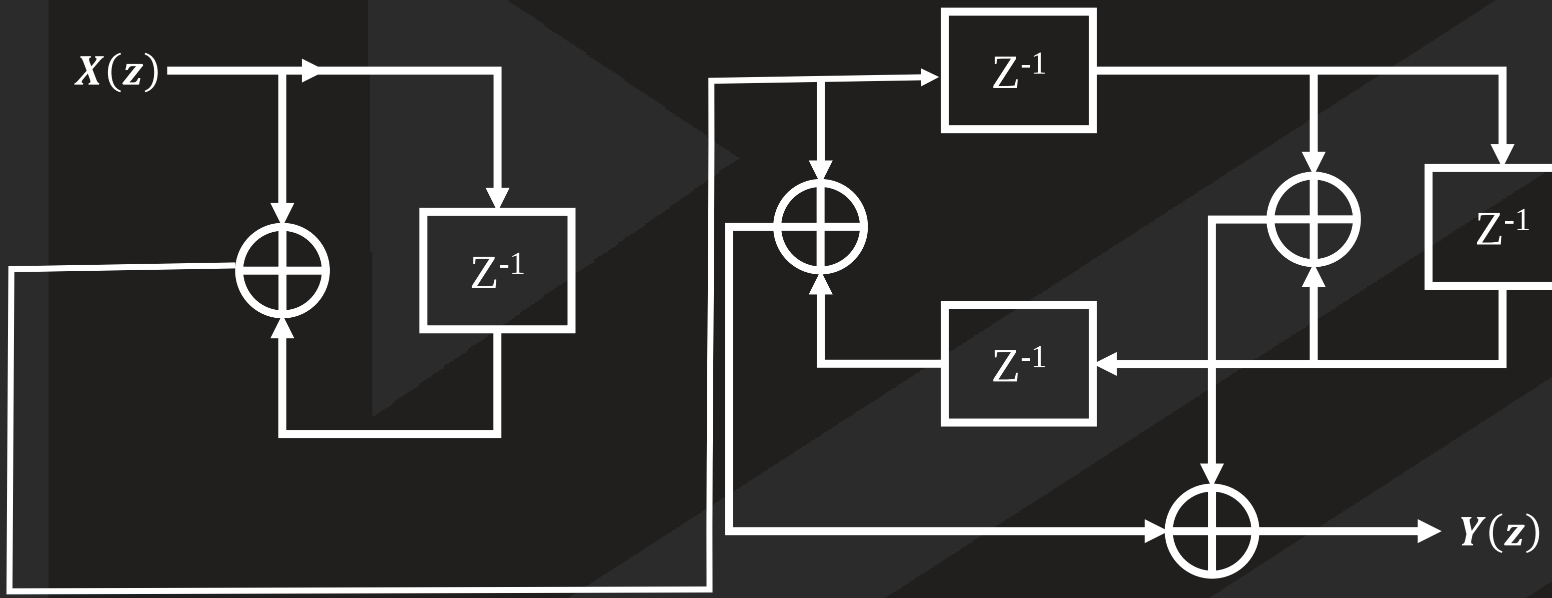
$+ \frac{1}{4}(z^{-2}+z^{-3})$



Linear phase realization

Q) Realise the system function using minimum number of multiplier

$$H(z) = (1 + z^{-1})\left(1 + \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} + z^{-3}\right)$$



Conversion of lattice coefficient to direct form filter coefficient

The general equation

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k)x(n-k)$$

The equations to convert filter coefficients to direct form FIR filter coefficients are

$$\alpha_m(0) = 1$$

$$\alpha_m(m) = K_m$$

$$\alpha_m(k) = \alpha_{m-1}(k) + \alpha_m(m)\alpha_{m-1}(m-k)$$

Q) Consider an FIR filter lattice filter with coefficient $K_1 = ½$,$K_2 = 1/3$, $K_3 = ¼$. Determine the FIR filter coefficients for direct form structure?

Solution

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k)x(n-k)$$

Number of stages (m) = 3

$$y(n) = x(n) + \sum_{k=1}^{3} \alpha_m(k)x(n-k) \qquad = x(n) + \alpha_3(1)x(n-1) + \alpha_3(2)x(n-2) + \alpha_3(3)x(n-3)$$

$$\alpha_m(0) = 1$$

$$\alpha_m(m) = K_m$$

$$\alpha_m(k) = \alpha_{m-1}(k) + \alpha_m(m)\alpha_{m-1}(m-k)$$

For m=3

$$\alpha_3(0) = 1$$

$$\alpha_3(3) = \frac{1}{4}$$

and k=2

$$\alpha_3(2) = \alpha_2(2) + \alpha_3(3)\alpha_2(1)$$

$$= \frac{1}{3} + \frac{1}{4}\alpha_2(1) \qquad = \frac{1}{2}$$

For m=2

$$\alpha_2(0) = 1$$

$$\alpha_2(2) = \frac{1}{3}$$

and k=1

$$\alpha_2(1) = \alpha_1(1) + \alpha_2(2)\alpha_1(1)$$

$$= \frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} \qquad = \frac{2}{3}$$

For m=1

$$\alpha_1(0) = 1$$

$$\alpha_1(1) = \frac{1}{2}$$

For m=3 and k=1

$$\alpha_3(1) = \alpha_2(1) + \alpha_3(3)\alpha_2(2)$$

$$= \frac{2}{3} + \frac{1}{4}\cdot\frac{1}{3} \qquad = \frac{3}{4}$$

Conversion of direct form FIR filter coefficient to lattice coefficient

The general equation

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k)x(n-k)$$

The equations to convert filter coefficients to lattice form FIR filter coefficients are

$$\alpha_{m-1}(0) = 1$$

$$K_m = \alpha_m(m)$$

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - \alpha_m(m)\alpha_m(m-k)}{1 - \alpha_m^2(m)} \quad for\ 1 \le k \le m-1$$

A FIR filter is given by the difference equation $y(n) = 2x(n) + \frac{4}{5}x(n-1) + \frac{3}{2}x(n-2) + \frac{2}{3}x(n-3)$ Determine its lattice form

Solution

$$y(n) = x(n) + \sum_{k=1}^{3} \alpha_m(k)x(n-k)$$

$$= x(n) + \alpha_3(1)x(n-1) + \alpha_3(2)x(n-2) + \alpha_3(3)x(n-3)$$

$$y(n) = 2\left[x(n) + \frac{2}{5}x(n-1) + \frac{3}{4}x(n-2) + \frac{1}{3}x(n-3)\right]$$

Comparing the two equations we get

$$\alpha_3(1) = \frac{2}{5} \qquad \alpha_3(2) = \frac{3}{4} \qquad \alpha_3(3) = \frac{1}{3} = k_3$$

$$\alpha_2(0) = 1$$

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k)x(n-k)$$

$$\alpha_{m-1}(0) = 1$$

$$K_m = \alpha_m(m)$$

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - \alpha_m(m)\alpha_m(m-k)}{1-\alpha_m^2(m)} \quad for\ 1 \le k \le m-1$$

For m=3, k=1

$\alpha_3(1) = \frac{2}{5}$ $\alpha_3(2) = \frac{3}{4}$ $\alpha_3(3) = \frac{1}{3} = k_3$ $\alpha_2(0) = 1$

$$\alpha_2(1) = \frac{\alpha_3(1) - \alpha_3(3)\alpha_3(2)}{1 - \alpha_3^2(3)} = \frac{\frac{2}{5} - \frac{1}{3}\cdot\frac{3}{4}}{1 - \left(\frac{1}{3}\right)^2} = 0.1687$$

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k)x(n-k)$$

For m=3, k=2

$$k_2 = \alpha_2(2) = \frac{\alpha_3(2) - \alpha_3(3)\alpha_3(1)}{1 - \alpha_2^2(3)} = \frac{\frac{3}{4} - \frac{1}{3}\cdot\frac{2}{5}}{1 - \left(\frac{1}{3}\right)^2} = 0.6937$$

For m=2, k=1

$$k_1 = \alpha_1(1) = \frac{\alpha_2(1) - \alpha_2(2)\alpha_2(1)}{1 - \alpha_2^2(2)} = \frac{0.1687 - (0.6937)0.1687}{1 - (0.6937)^2}$$

$$= 0.0996$$

$$\alpha_{m-1}(0) = 1$$

$$K_m = \alpha_m(m)$$

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - \alpha_m(m)\alpha_m(m-k)}{1 - \alpha_m^2(m)} \quad for\ 1 \le k \le m-1$$

IIR filter Realisation

- Mainly four types of realisation are there
  1. Direct form I realisation
  2. Direct form II realisation
  3. Cascade form realisation
  4. Parallel form realisation

Direct form I realization

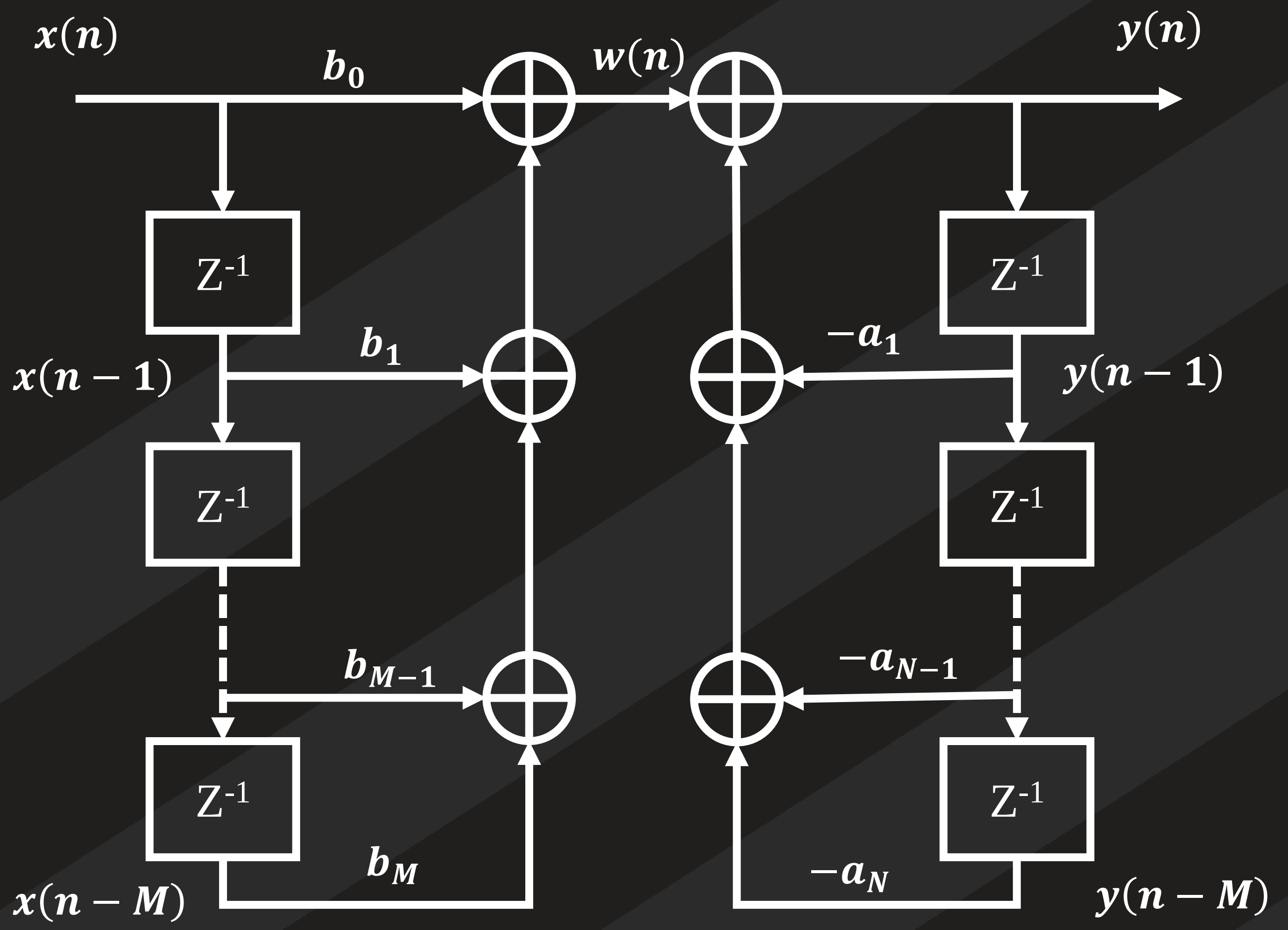
Let us consider an IIR system described by the difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$= -a_1 y(n-1) - a_2 y(n-2) \ldots - a_N y(n-N) + w(n)$$

Where

$$w(n) = b_0 x(n) + b_1 x(n-N) + \cdots + b_M x(n-M)$$



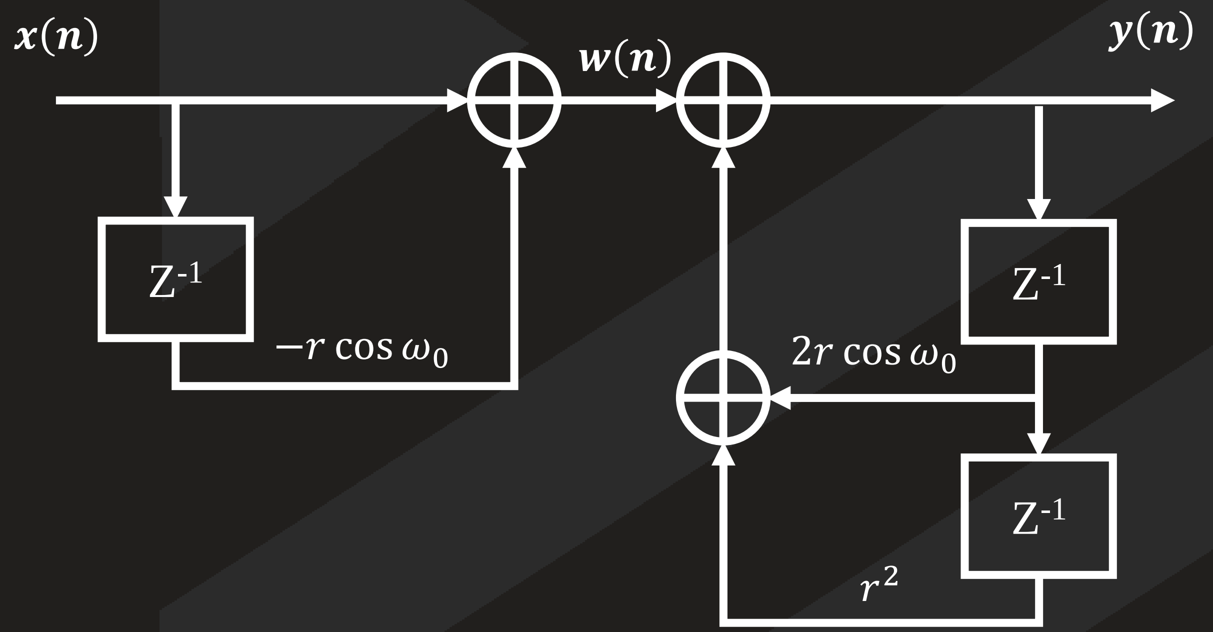
Direct form I realization

Q) Realise the second order digital filter

$$y(n) = 2r\cos\omega_0\, y(n-1) - r^2 y(n-2) + x(n) - r\cos\omega_0\, x(n-1)$$

Solution

$$w(n) = x(n) - r\cos\omega_0\, x(n-1) \qquad y(n) = 2r\cos\omega_0\, y(n-1) - r^2 y(n-2) + w(n)$$



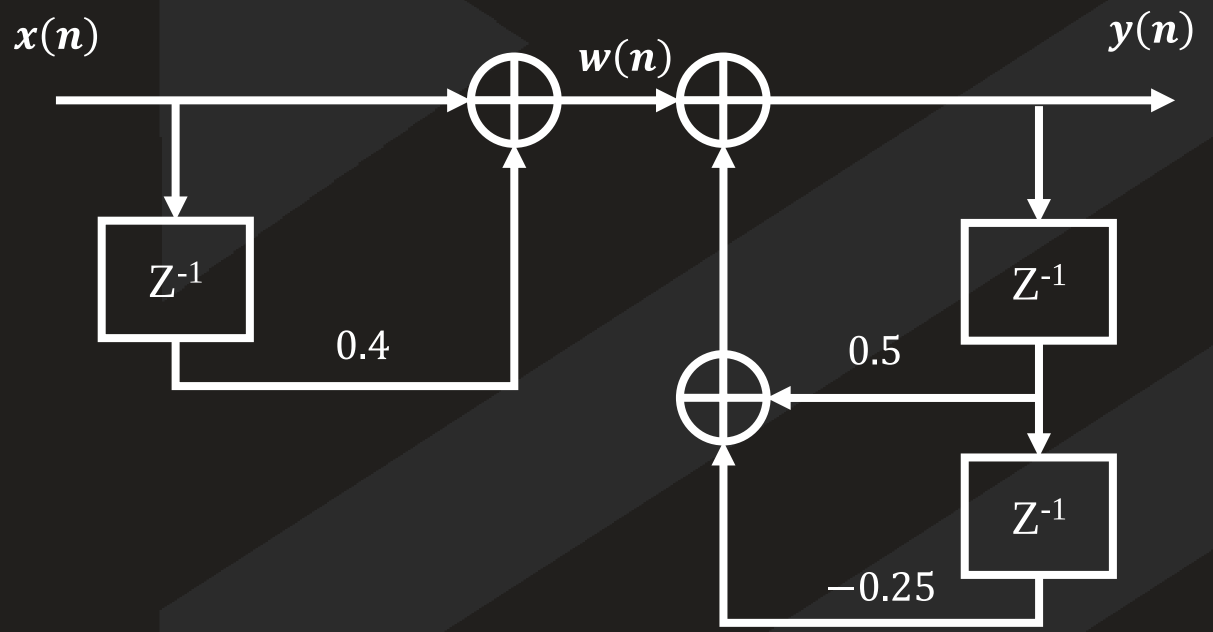
Direct form I realization

Q) Obtain the direct form I realisation for the system described by difference equation

$$y(n) = 0.5y(n-1) - 0.25y(n-2) + x(n) + 0.4x(n-1)$$

Solution

$$w(n) = x(n) + 0.4x(n-1) \qquad y(n) = 0.5y(n-1) - 0.25y(n-2) + w(n)$$



Direct form II realization

Let us consider an IIR system described by the difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

The system function can be represented as

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Let

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

$$\frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + \cdots + b_M z^{-M} W(z)$$

where

$$W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) \ldots + a_N z^{-N} W(z) = X(z)$$

$$W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) \ldots - a_N z^{-N} W(z)$$

Taking inverse z transform we get

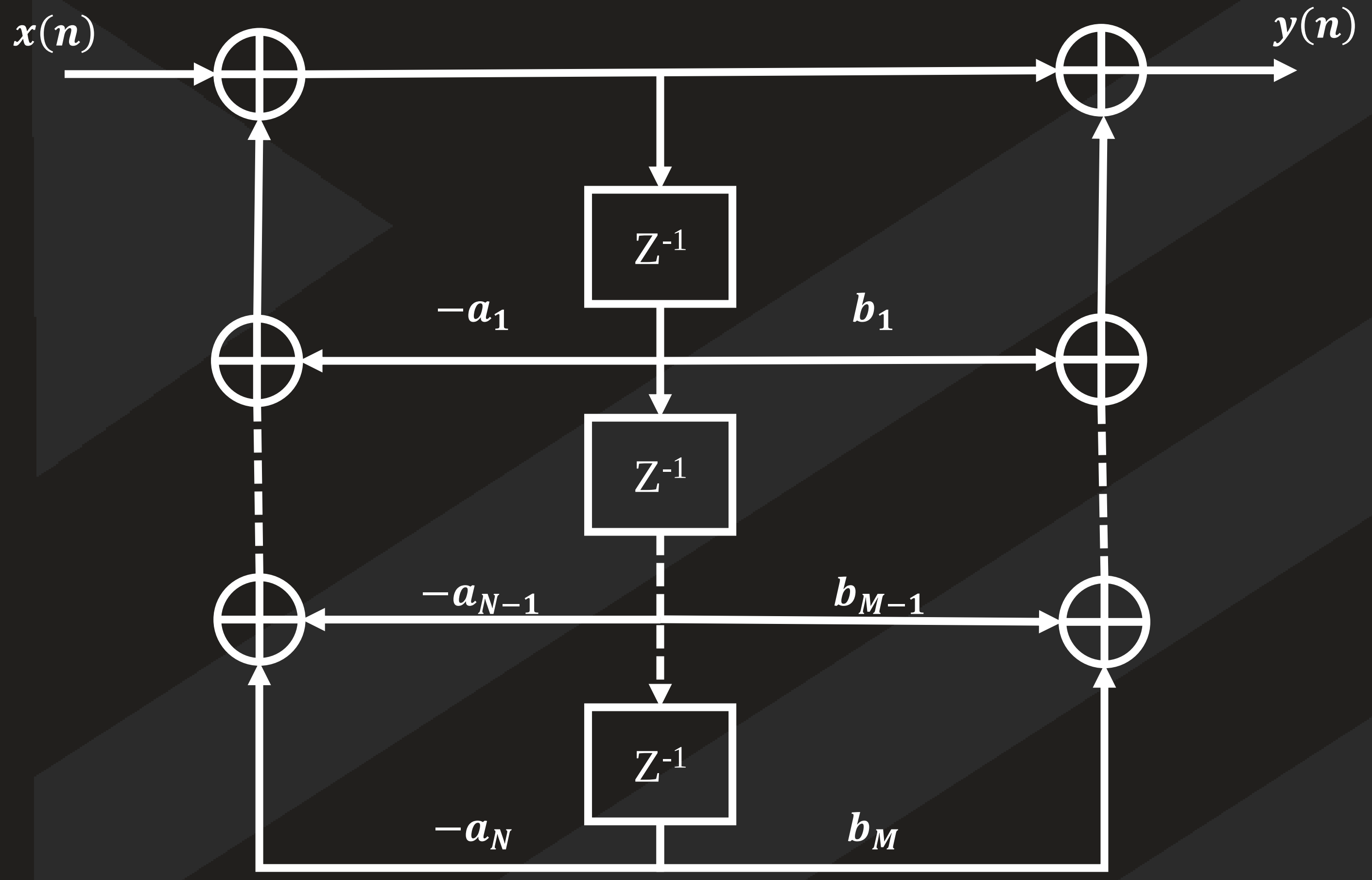
$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) \ldots - a_N w(n-N)$$

$$y(n) = b_0 w(n) + b_1 w(n-1) + \cdots + b_M w(n-M)$$

Direct form II realization

$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) \ldots - a_N w(n-N)$$

$$y(n) = b_0 w(n) + b_1 w(n-1) + \cdots + b_M w(n-M)$$



Direct form II realization

Q) Realise the second order digital filter using direct form II

$$y(n) = 2r\cos\omega_0\, y(n-1) - r^2 y(n-2) + x(n) - r\cos\omega_0\, x(n-1)$$

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

Solution

Taking Z - Transform

$$Y(z) = 2r\cos\omega_0\, Y(z)z^{-1} - r^2 Y(z)z^{-2} + X(z) - r\cos\omega_0\, X(z)z^{-1}$$

$$Y(z)[1 - 2r\cos\omega_0\, z^{-1} + r^2 z^{-2}] = X(z)[1 - r\cos\omega_0\, z^{-1}]$$

$$\frac{Y(z)}{X(z)} = \frac{1 - r\cos\omega_0\, z^{-1}}{1 - 2r\cos\omega_0\, z^{-1} + r^2 z^{-2}}$$

$$\frac{Y(z)}{W(z)} = 1 - r\cos\omega_0\, z^{-1}$$

$$Y(z) = W(z)[1 - r\cos\omega_0\, z^{-1}]$$

Inverse transform

$$y(n) = w(n) - r\cos\omega_0\, w(n-1)$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 - 2r\cos\omega_0\, z^{-1} + r^2 z^{-2}}$$

$$W(z) - 2r\cos\omega_0\, W(z)z^{-1} + r^2 W(z)z^{-2} = X(z)$$

$$W(z) = X(z) + 2r\cos\omega_0\, z^{-1}W(z) - r^2 z^{-2} W(z)$$

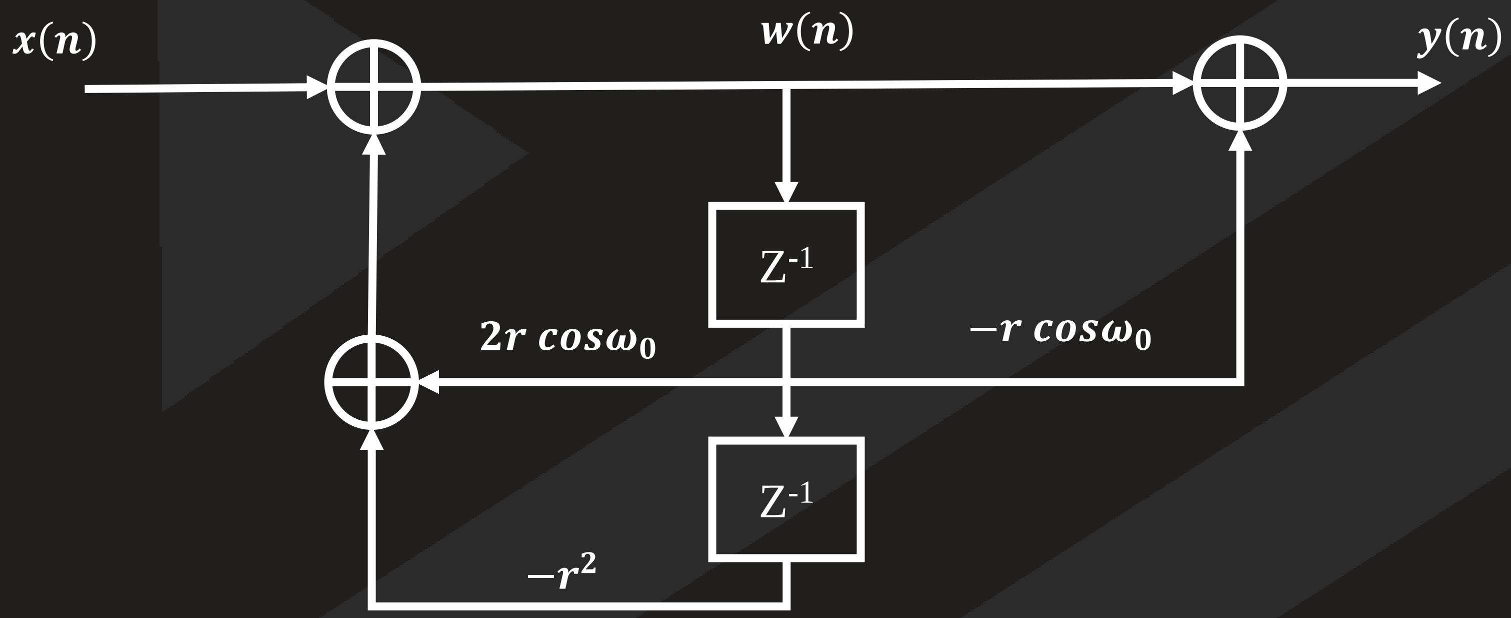
Inverse transform

$$w(n) = x(n) + 2r\cos\omega_0\, x(n-1) - r^2 x(n-2)$$

Direct form II realization

$y(n) = w(n) - r\cos\omega_0\, w(n-1)z^{-1}$

$w(n) = x(n) + 2r\cos\omega_0\, x(n-1) - r^2 x(n-2)$



Direct form II realization

Q) Determine the direct form II realisation for the following system

$$y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-1)$$

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)}$$

Solution

Taking Z - Transform

$$Y(z) = -0.1Y(z)z^{-1} + 0.72Y(z)z^{-2} + 0.7X(z) - 0.252X(z)z^{-1}$$

$$Y(z)[1 + 0.1z^{-1} - 0.72z^{-2}] = X(z)[0.7 - 0.252z^{-1}]$$

$$\frac{Y(z)}{X(z)} = \frac{0.7 - 0.252z^{-1}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$\frac{Y(z)}{W(z)} = 0.7 - 0.252z^{-1}$$

$$Y(z) = W(z)[0.7 - 0.252z^{-1}]$$

Inverse transform

$$y(n) = 0.7w(n) - 0.252w(n-1)$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$W(z) + 0.1W(z)z^{-1} - 0.72W(z)z^{-2} = X(z)$$

$$W(z) = X(z) - 0.1W(z)z^{-1} + 0.72W(z)z^{-2}$$

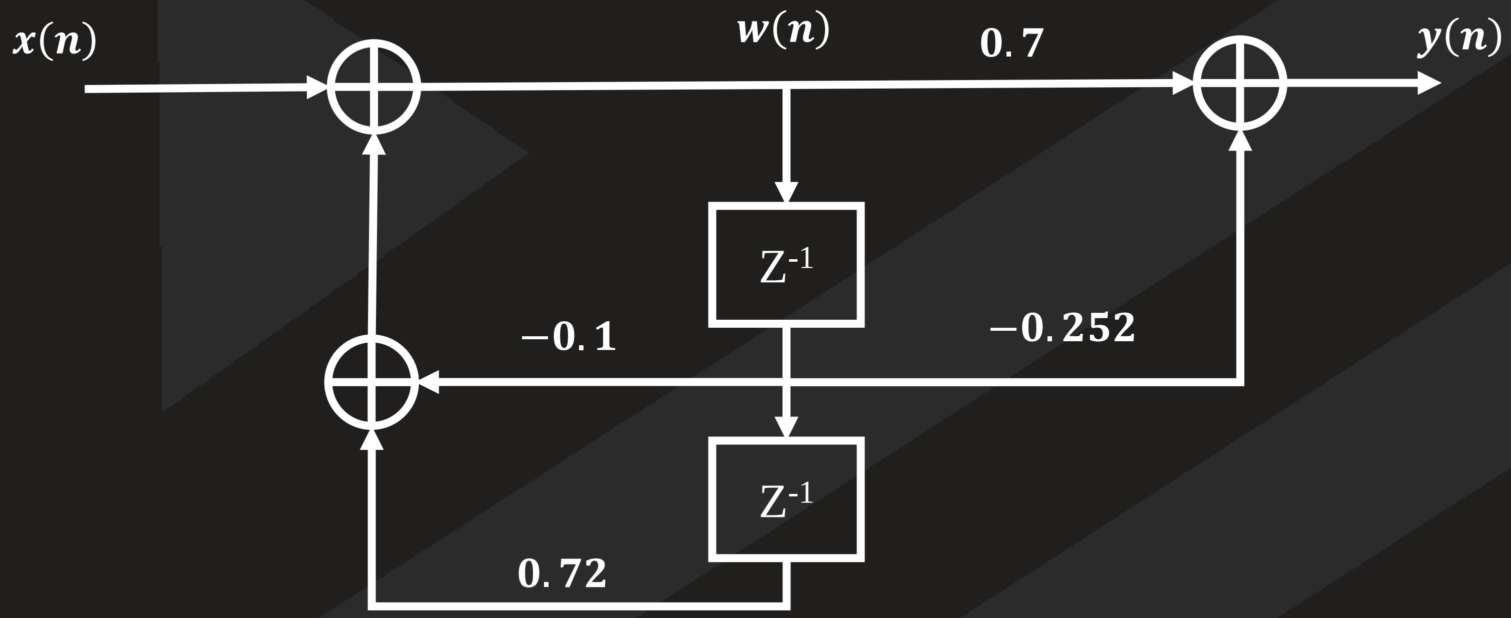
Inverse transform

$$w(n) = x(n) - 0.1w(n-1) + 0.72w(n-2)$$

Direct form II realization

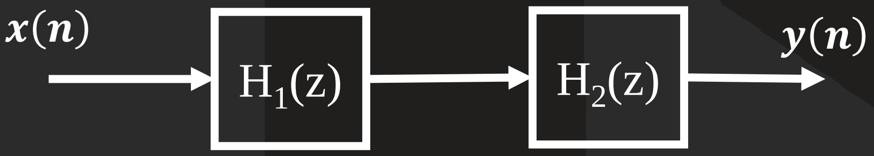
$y(n) = 0.7w(n) - 0.252w(n-1)$

$w(n) = x(n) - 0.1w(n-1) + 0.72w(n-2)$



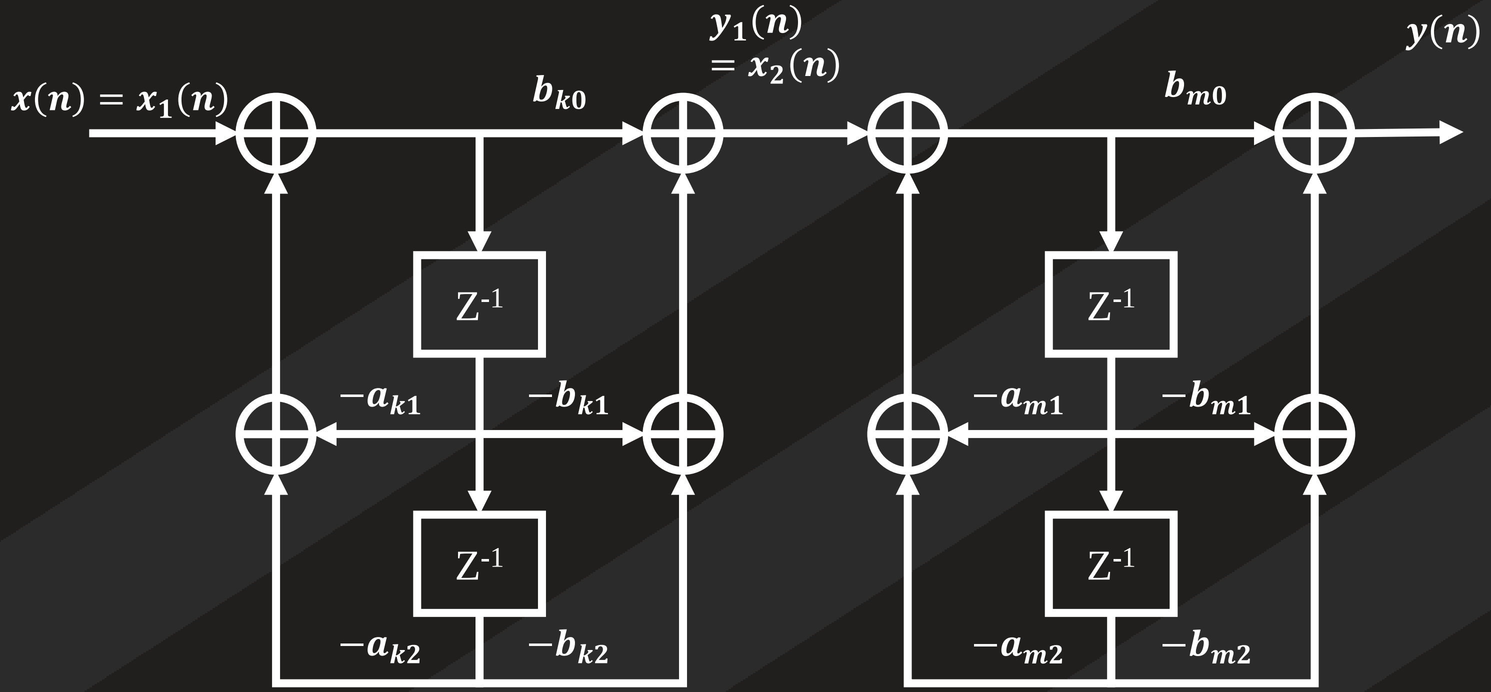
Cascade form realization

Let us consider an IIR system with system function



$$H_1(z) = \frac{b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}$$

$$H_2(z) = \frac{b_{m0} + b_{m1}z^{-1} + b_{m2}z^{-2}}{1 + a_{m1}z^{-1} + a_{m2}z^{-2}}$$



Cascade form realization

Q) Realise the system with difference equation

$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n) + \frac{1}{3}x(n-1)$ *in cascade form*

$$H(z) = \frac{Y(z)}{X(z)}$$

Solution

Taking Z - Transform

$$Y(z) = \frac{3}{4}Y(z)z^{-1} - \frac{1}{8}Y(z)z^{-2} + X(z) + \frac{1}{3}X(z)z^{-1}$$

$$Y(z)\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] = X(z)\left[1 + \frac{1}{3}z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$\frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{\left[1 - \frac{1}{2}z^{-1}\right]\left[1 - \frac{1}{4}z^{-1}\right]}$$

$$H_1(z) = \frac{1 + \frac{1}{3}z^{-1}}{\left[1 - \frac{1}{2}z^{-1}\right]}$$

$$H_2(z) = \frac{1}{\left[1 - \frac{1}{4}z^{-1}\right]}$$

YouTube - IMPLearn

$$H_1(z) = \frac{1 + \frac{1}{3}z^{-1}}{\left[1 - \frac{1}{2}z^{-1}\right]}$$

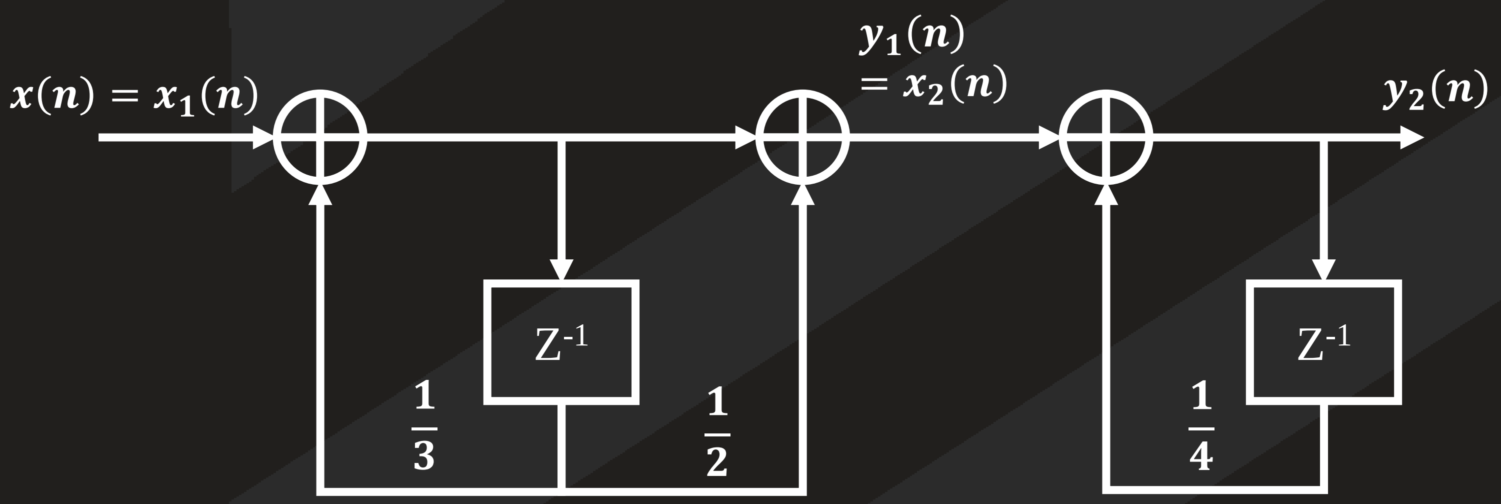
$$H_2(z) = \frac{1}{\left[1 - \frac{1}{4}z^{-1}\right]}$$

$$Y_1(z) = X_1(z) + \frac{1}{3}X_1(z)z^{-1} + \frac{1}{2}Y_1(z)z^{-1}$$

$$y_1(n) = x_1(n) + \frac{1}{3}x_1(n-1) + \frac{1}{2}y_1(n-1)$$

$$Y_2(z) = X_2(z) + \frac{1}{4}Y_2(z)$$

$$y_2(n) = x_2(n) + \frac{1}{4}y_2(n-1)$$



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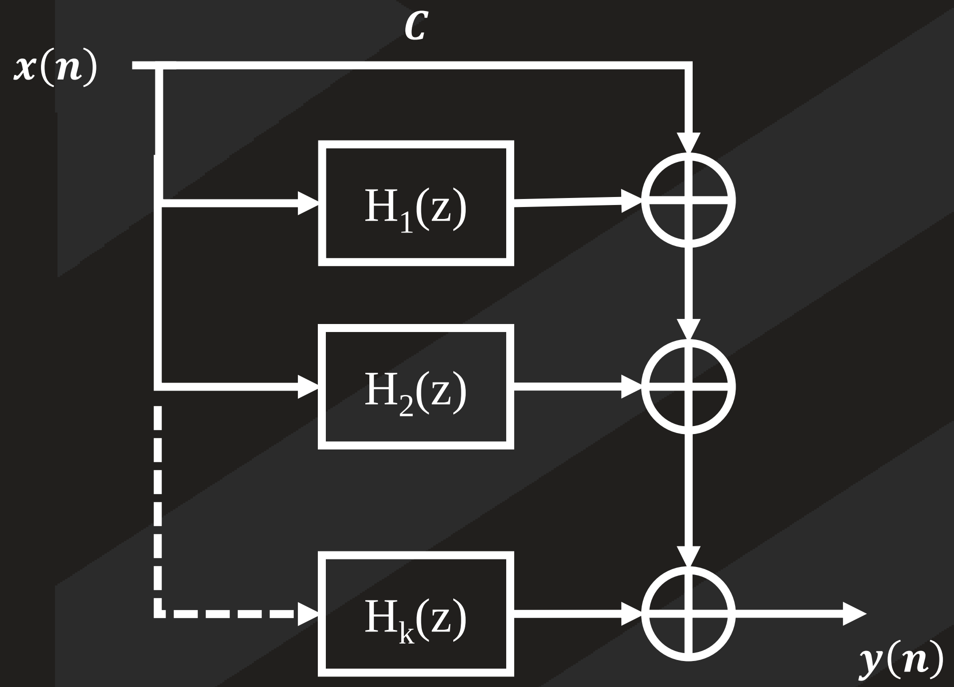
Parallel form realization

A parallel form realisation of an IIR system can be obtained by performing a partial expansion

$$H(z) = C + \sum_{k=1}^{N} \frac{C_k}{1 - P_k z^{-1}} \qquad H(z) = C + \frac{C_1}{1 - P_1 z^{-1}} + \frac{C_2}{1 - P_2 z^{-1}} + \cdots + \frac{C_N}{1 - P_N z^{-1}}$$

$$Y(z) = cX(z) + H_1(z)X(z) + H_2(z)X(z) + \cdots + H_N(z)X(z)$$



Parallel form realization

Q) Realise the system given by difference equation

$y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.25x(n-2)$ *in parallel form*

Taking Z - Transform

$$Y(z) = -0.1Y(z)z^{-1} + 0.72Y(z)z^{-2} + 0.7X(z) - 0.25X(z)z^{-2}$$

$$Y(z)[1 + \ 0.1z^{-1} - 0.72z^{-2}] = X(z)[0.7 - 0.25z^{-2}]$$

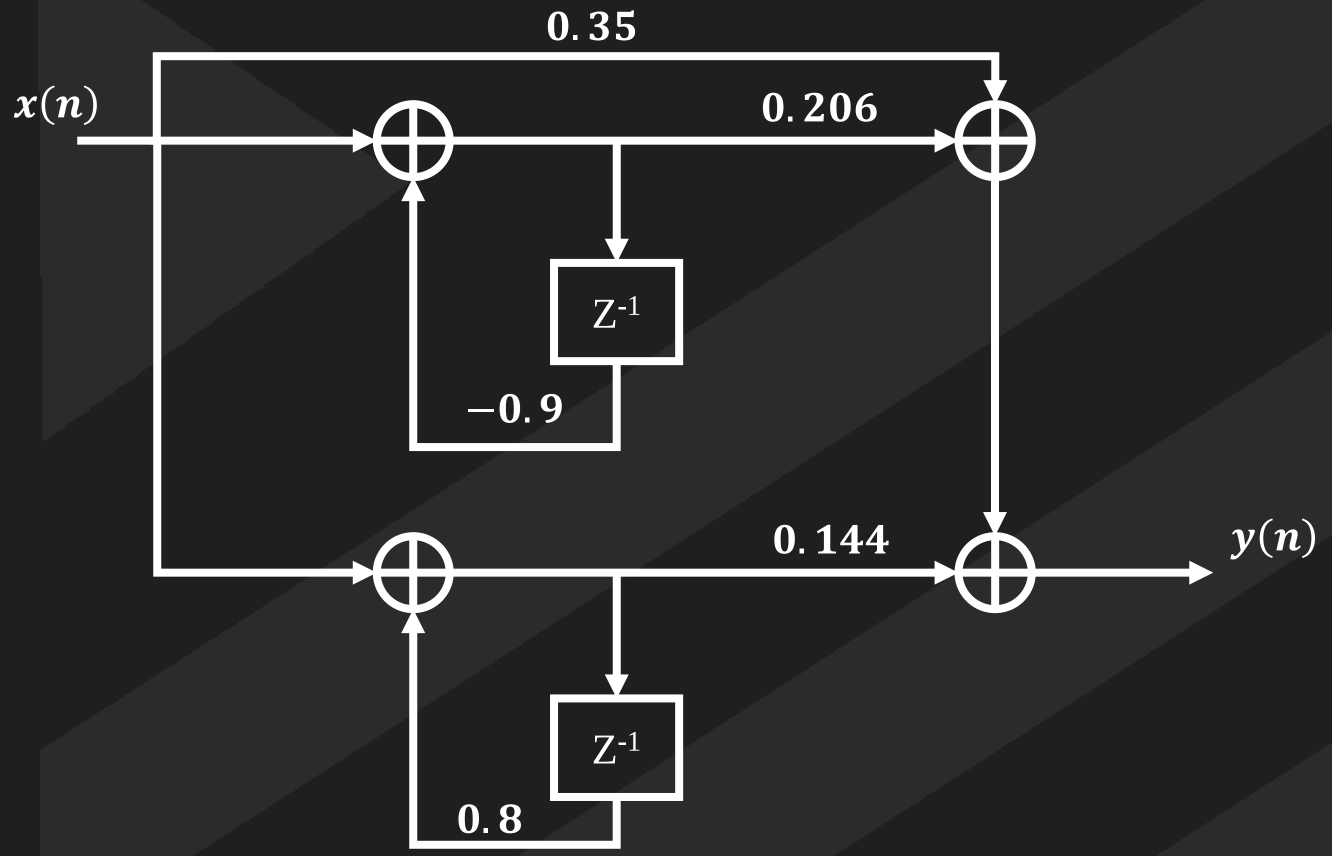
$$H(z) = \frac{0.7-0.25z^{-2}}{1+\ 0.1z^{-1}-0.72z^{-2}}$$

$$= 0.35 + \frac{-0.035z^{-1}+0.35}{1+0.1z^{-1}-0.72z^{-2}}$$

$$= 0.35 + \frac{0.206}{1+0.9z^{-1}} + \frac{0.144}{1-0.8z^{-1}}$$

Parallel form realization

$$H(z) = 0.35 + \frac{0.206}{1 + 0.9z^{-1}} + \frac{0.144}{1 - 0.8z^{-1}}$$

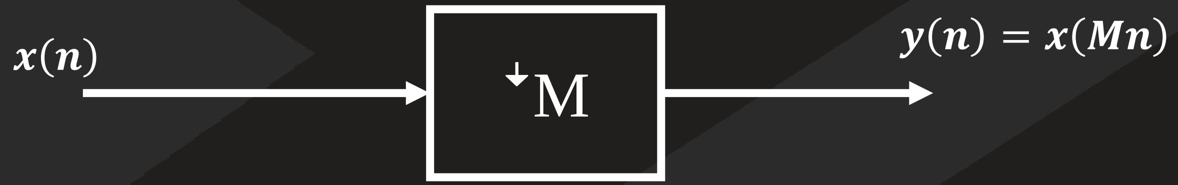


Multi rate DSP

- The processing of a discrete time signal at different sampling rates in different parts of a system is called multi rate DSP
- Discrete time system that employ sampling rate conversion while processing the discrete time signal are called multi rate DSP system
- The process of converting a signal from one sampling rate to another sampling rate are of two types
  - Down sampling or decimation
  - Up sampling or interpolation

Down sampling

- It is the process of reducing the sampling rate by an integer factor D or M
- Down sampled signal of *x(n)* can be obtained by simply keeping every *M*th sample and removing *(M-1)* in between samples



$x(n) = \{1, -1, 2, 4, 0, 3, 2, 1, 5\}$ *for M=2*

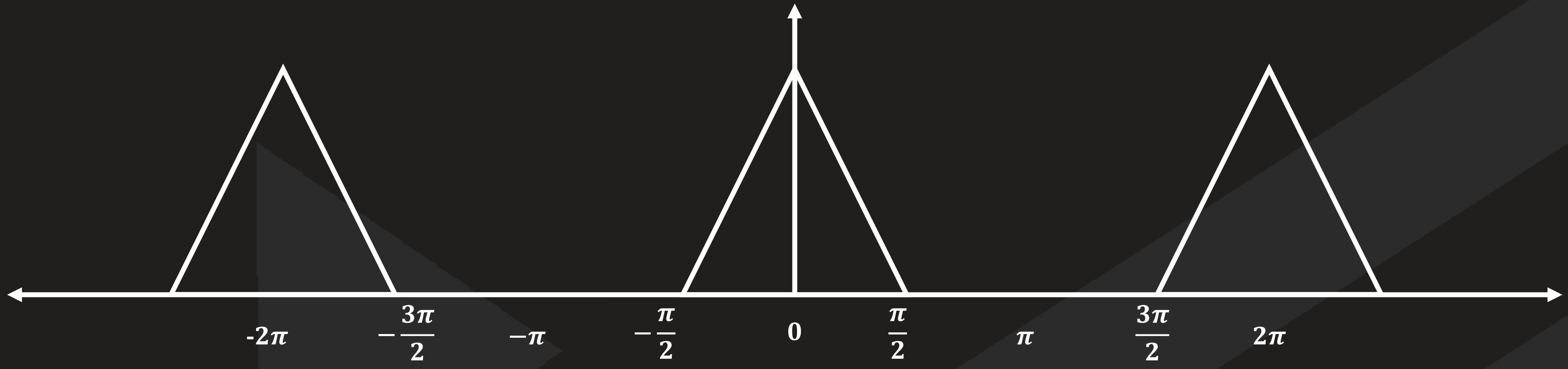
$x(Mn) = \{1, 2, 0, 2, 5\}$

Spectrum of the down sampled signal

$$Y\left(e^{j\omega}\right) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j\left(\frac{\omega-2\pi k}{M}\right)}\right)$$

- If the Fourier transform of the input-signal of a down samples in $X\left(e^{j\omega}\right)$, then the Fourier transform Y$\left(e^{j\omega}\right)$ of the output signal y(n) is a sum of M uniformly shifted and stretched version of $X\left(e^{j\omega}\right)$ scaled by a factor of 1/M

Q) Consider a spectrum of input signal $X(e^{j\omega})$ with a bandwidth of $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ shown , when the signal is down sampled by a factor D, sketch the spectrum of a down sampled signal for sampling rate reduction factor D=2,3



For M/D=2

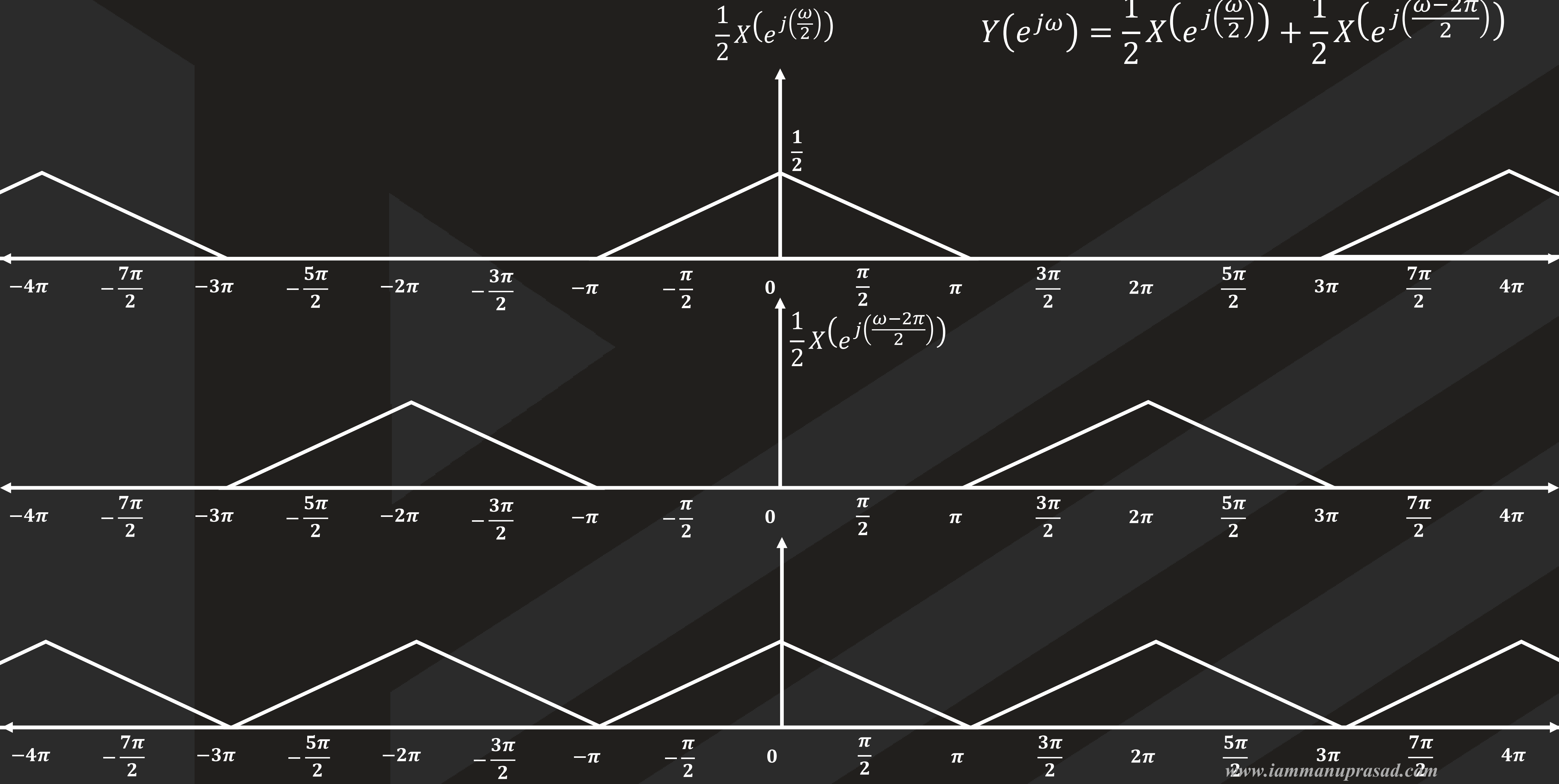
$$Y(e^{j\omega}) = \frac{1}{2}\sum_{k=0}^{1} X\left(e^{j\left(\frac{\omega-2\pi k}{2}\right)}\right)$$

$$Y(e^{j\omega}) = \frac{1}{2}X\left(e^{j\left(\frac{\omega}{2}\right)}\right) + \frac{1}{2}X\left(e^{j\left(\frac{\omega-2\pi}{2}\right)}\right)$$

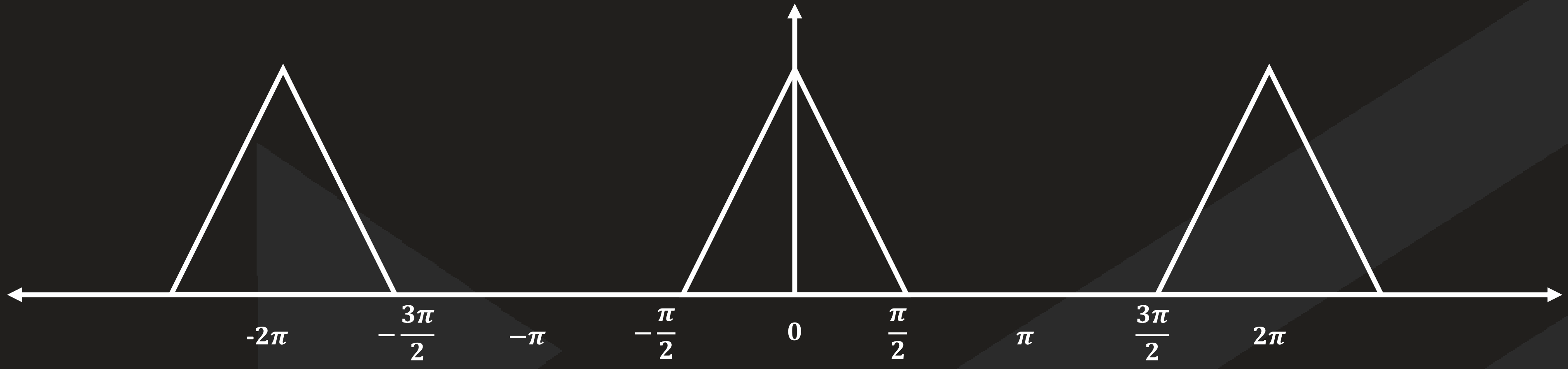
When D=2

Bandwidth =D.BW =2π

$$Y(e^{j\omega}) = \frac{1}{2}X\left(e^{j\left(\frac{\omega}{2}\right)}\right) + \frac{1}{2}X\left(e^{j\left(\frac{\omega-2\pi}{2}\right)}\right)$$



Q) Consider a spectrum of input signal $X(e^{j\omega})$ with a bandwidth of $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ shown , when the signal is down sampled by a factor D, sketch the spectrum of a down sampled signal for sampling rate reduction factor D=2,3



For M/D=3

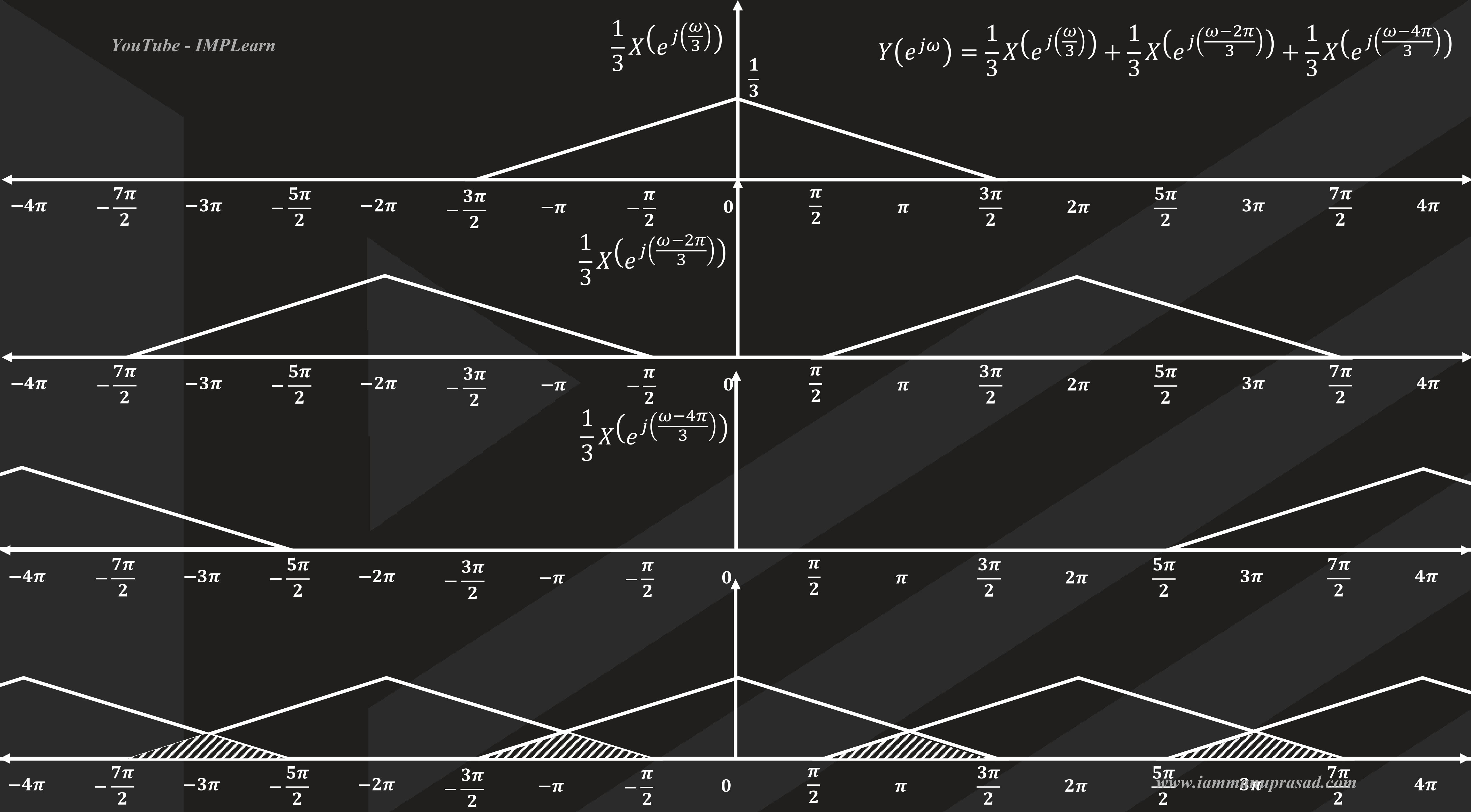
When D=3

Bandwidth =D.BW =3π

$$Y(e^{j\omega}) = \frac{1}{3}\sum_{k=0}^{2} X\left(e^{j\left(\frac{\omega-2\pi k}{3}\right)}\right)$$

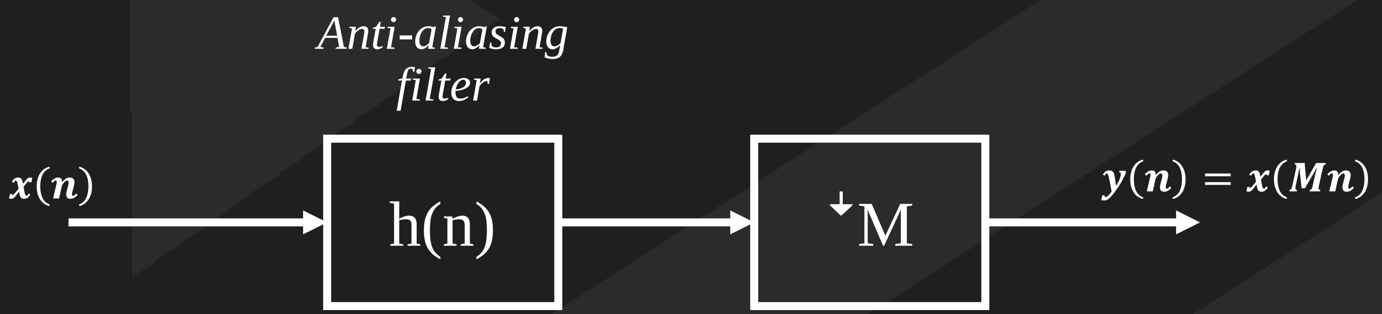
$$Y(e^{j\omega}) = \frac{1}{3}X\left(e^{j\left(\frac{\omega}{3}\right)}\right) + \frac{1}{3}X\left(e^{j\left(\frac{\omega-2\pi}{3}\right)}\right) + \frac{1}{3}X\left(e^{j\left(\frac{\omega-4\pi}{3}\right)}\right)$$

$$Y(e^{j\omega}) = \frac{1}{3}X\left(e^{j\left(\frac{\omega}{3}\right)}\right) + \frac{1}{3}X\left(e^{j\left(\frac{\omega-2\pi}{3}\right)}\right) + \frac{1}{3}X\left(e^{j\left(\frac{\omega-4\pi}{3}\right)}\right)$$



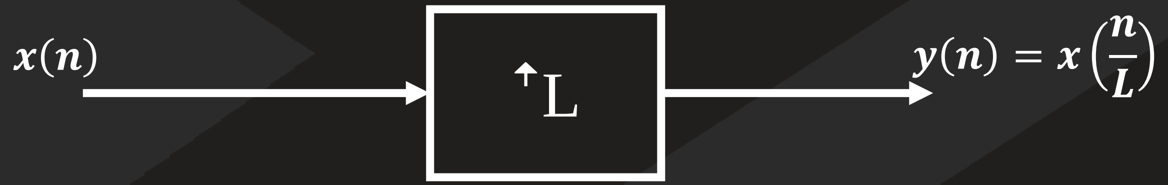
Anti-aliasing filters

- *In order to avoid aliasing the input signal should be band limited to π/D for decimation by a factor of D*



Up sampling

- *It is the process of increasing the sampling rate by an integer factor I or L*
- *Up sampled signal of x(n) can be obtained by a factor of L by L-1 equally spaced zeros between each pairs of samples*



$x(n) = \{1, 2, 3, 4, 5\}$ *for I/L=3*

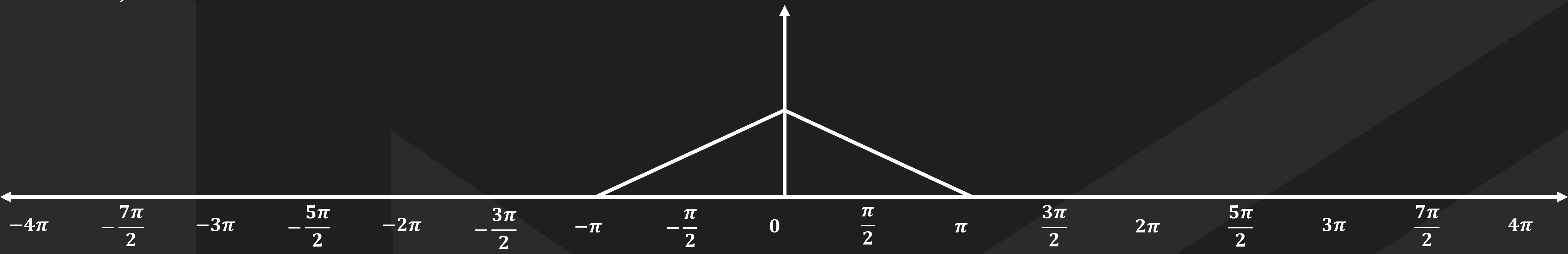
$x\left(\frac{n}{L}\right) = \{1, 0, 0, 2, 0, 0, 3, 0, 0, 4,0,0\ \}$

Spectrum of the up sampled signal

$$Y(e^{j\omega}) = X(e^{j\omega I})$$

- The term $X(e^{j\omega I})$ is the frequency compressed version of $X(e^{j\omega})$ by a factor I.
- If the frequency response is periodic with 2π , the $X(e^{j\omega I})$ will repeat I times in a period of 0 to 2π in the spectrum of up sampled

Q) The spectrum of discrete time signal is shown below. Draw the spectrum of the signal if it is upscaled by I=2,3

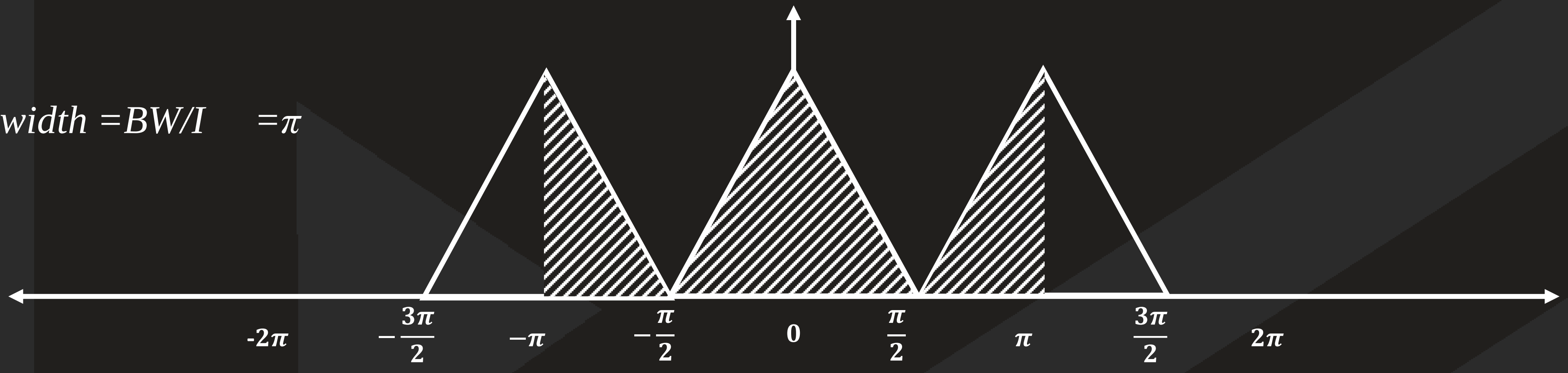


For I or L=2

$$Y\left(e^{j\omega}\right) = X\left(e^{j2\omega}\right)$$

When I=2

Bandwidth =BW/I =π

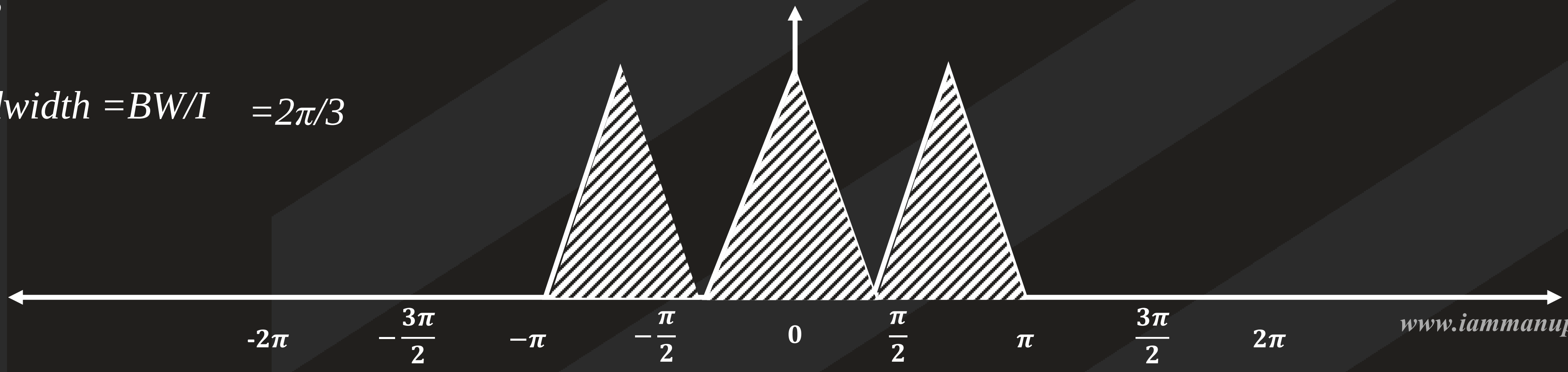


For I or L=3

$$Y\left(e^{j\omega}\right) = X\left(e^{j3\omega}\right)$$

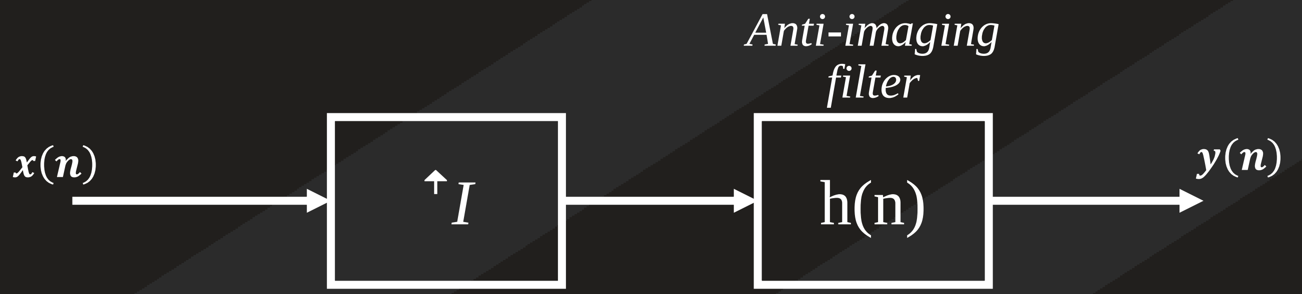
When I=3

Bandwidth =BW/I =2π/3



Anti-imaging filters

- *When up sampled by a factor of I, the output spectrum will have I images in each period with each image bandwidth to* $\frac{\pi}{I}$
- *Since the frequency spectrum in the range to 0 to* $\frac{\pi}{I}$ *are unique and we have to filter the other images*
- *Hence the output of up samples is passed through a lowpass filter with band width* $\frac{\pi}{I}$
- *Since the lowpass filter is designed to avoid multiple images in output spectrum , it also called anti-imaging filter*



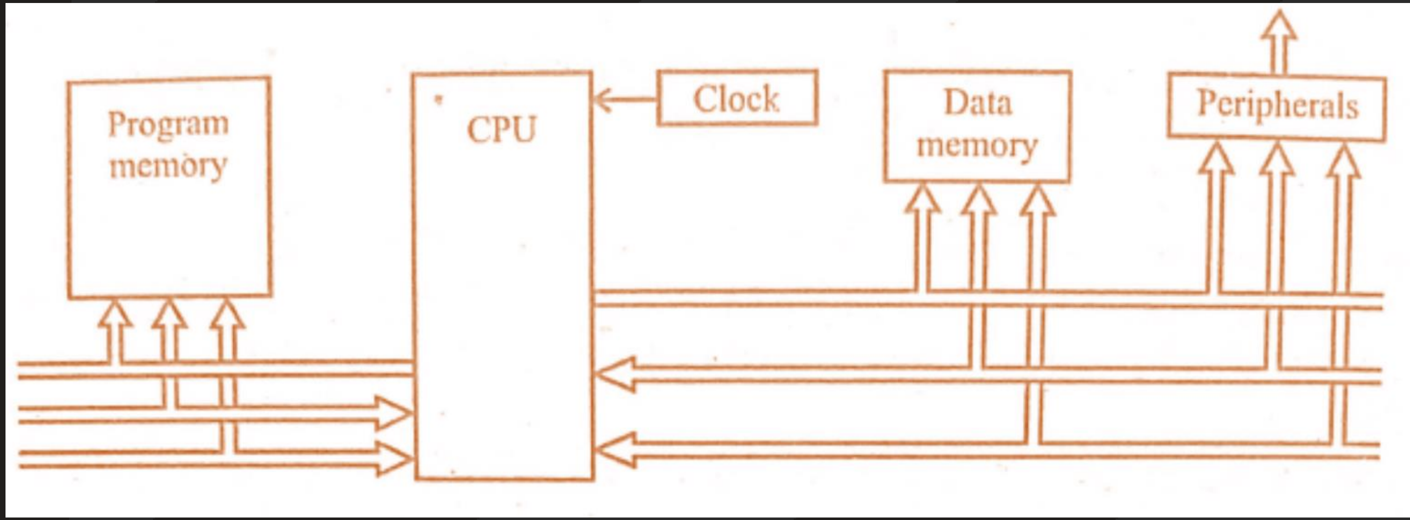
MODULE 5

Digital Signal Processors

- The programmable digital signal processors (PDSPs) are general purpose microprocessors designed specifically for digital signal processing applications.
- They contain special architecture and instruction set to execute computation - intensive DSP algorithms more efficiently.
    - ***General purpose digital signal processors***: These are basically high-speed microprocessors with architecture and instruction sets optimized for DSP operations.
    - ***Special purpose digital signal processors:*** These types of processors consist of hardware i) designed for specific DSP algorithms such as FFr, ii) hardware designed for specific applications such as PCM and filtering.

Harvard Architecture

- The term Harvard originated from the Harvard Mark 1 relay-based computer which stored instruction on punched tape and data in relay latches
- The Harvard architectures physically separate memories for their instructions and data, requiring dedicated buses for each of them.
- Instructions and operands can therefore be fetched simultaneously.
- Most of the DSP processors use a modified Harvard architecture with two or three memory buses; allowing access to filter coefficients and input signals in the same cycle.
- Since it possesses two independent bus systems, the Harvard architecture is capable of simultaneous reading an instruction code and reading or writing a memory or peripheral as part of the execution of the previous instruction.



Pipelining

- To improve the efficiency, advanced microprocessors and digital signal processors use an approach called pipelining in which different phases of operation and execution of instructions are carried out in parallel.
- In modem processors the first step of execution is performed on the first instruction, and then when the instruction passes to the next step, a new instruction is started.
  - ***The Fetch phase(F)*** in which the next instruction is fetched from the address stored in the program counter.
  - ***The decode phase (D)*** in which the instruction in the instruction register is decoded and the address in the program counter is incremented
  - ***Memory read (R)*** phase reads the data from the data buses and also writes data to the data buses.
  - ***The Execute phase (X)*** executes the instruction currently in the instruction register and also completes the write process.

Instruction 1	F_1	D_1	R_1	X_1			
Instruction 2		F_2	D_2	R_2	X_2		
Instruction 3			F_3	D_3	R_3	X_3	
Instruction 4				F_4	D_4	R_4	X_4

Multiply Accumulate Unit (MAC)

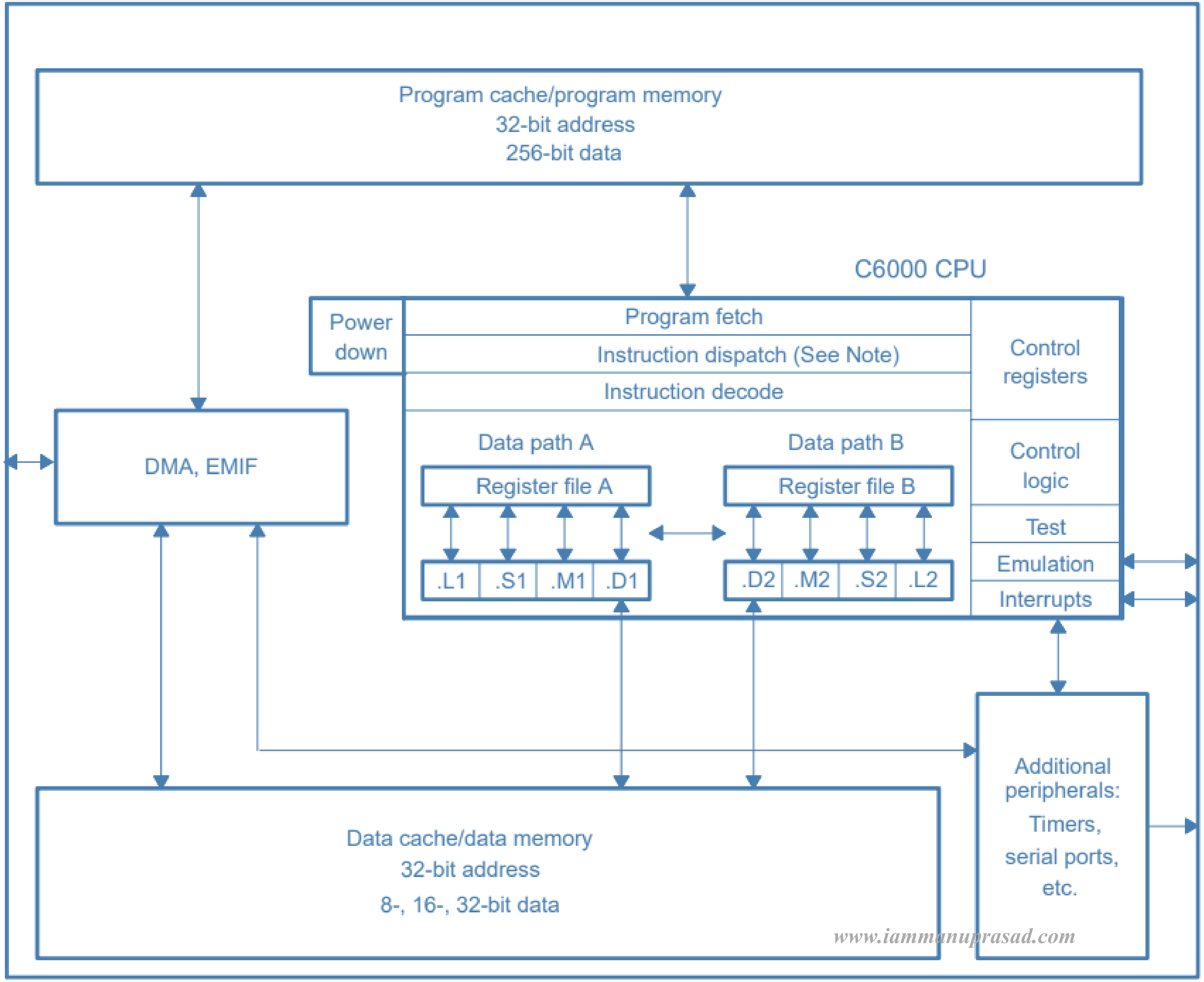
- The Multiply-Accumulate (MAC) operation is the basis of many digital signal processing algorithms
- In digital signal processing, the multiply–accumulate (MAC) operation is a common step that computes the product of two numbers and adds that product to an accumulator.
- The hardware unit that performs the operation is known as a multiplier–accumulator (MAC unit); the operation itself is also often called a MAC
- The MAC speed applies both to finite impulse response (FIR) and infinite impulse response (IIR) fi1ters. The complexity of the filter response dictates the number MAC operations required per sample period.
- A multiply-accumulate step performs the following:
  - Reads a 16-bit sample data (pointed to by a register)
  - Increments the sample data-pointer by 2
  - Reads a. 16-bit coefficient (pointed to by another register)
  - Increments the coefficient register pointer by 2
  - Sign Multiply (16-bit) data and coefficient 'to yield a 32~bit resu1t
  - Adds the result to the contents of a 32-bit register pair for accumulate.

TMS320C67xx - Digital Signal Processor

- The **TMS320 DSP** family consists of fixed-point, floating-point, and multiprocessor digital signal processors (DSPs).
- TMS320 DSPs have an architecture designed specifically for real-time signal processing.
- With a performance of up to 6000 million instructions per second (MIPS) and an efficient C compiler, the **TMS320C6000 DSPs** give system architects unlimited possibilities to differentiate their products.
  - High Performance
  - Ease of use
  - affordable pricing
- The C6000 devices execute up to eight 32-bit instructions per cycle. The C67x CPU consists of 32 general-purpose 32-bit registers and eight functional units.
- These eight functional units contain:
  - Two multipliers
  - Six ALUs

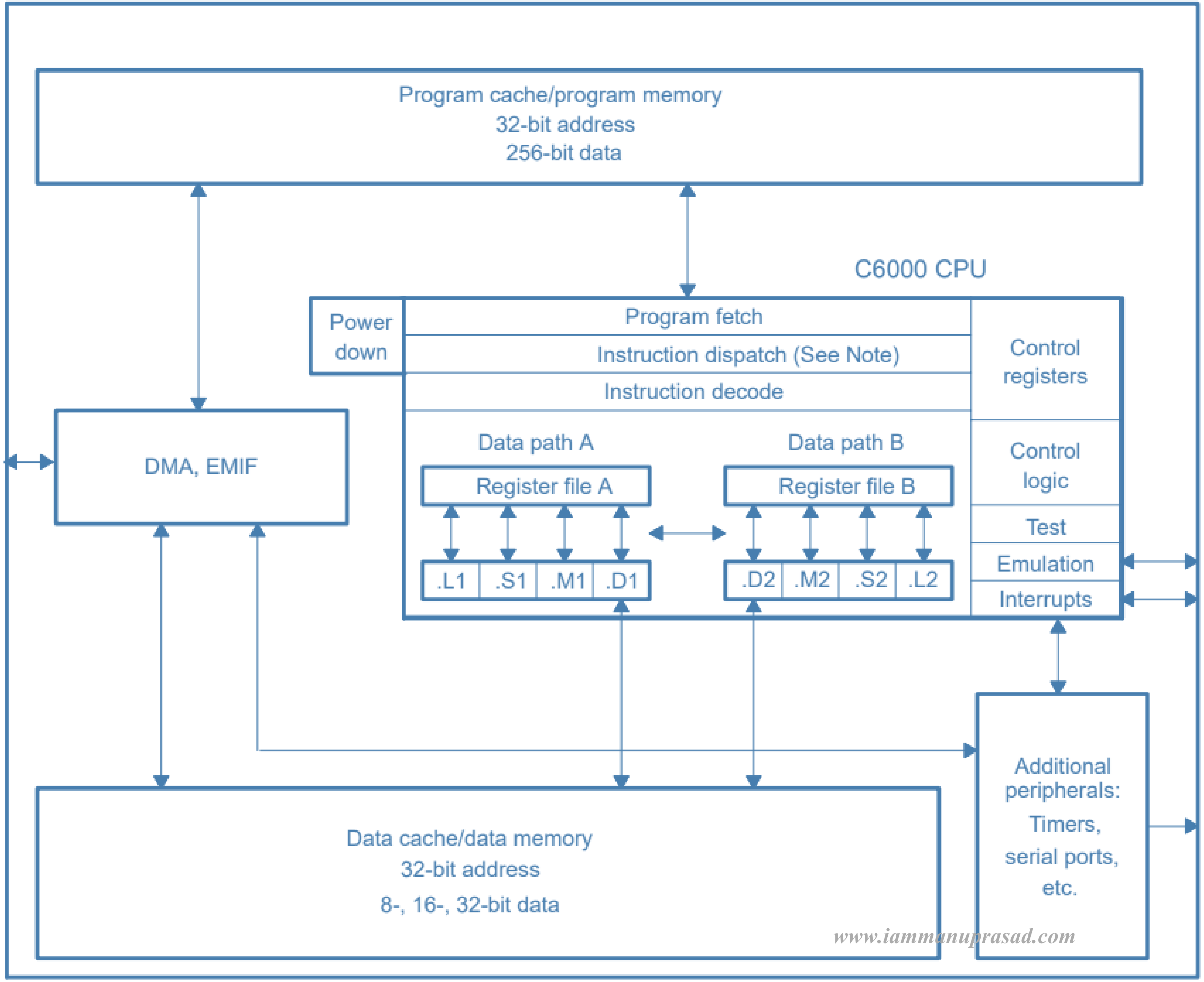
TMS320C67xx DSP Architecture

- Central Processing Unit (CPU)
- Internal Memory
- Memory and Peripheral Options



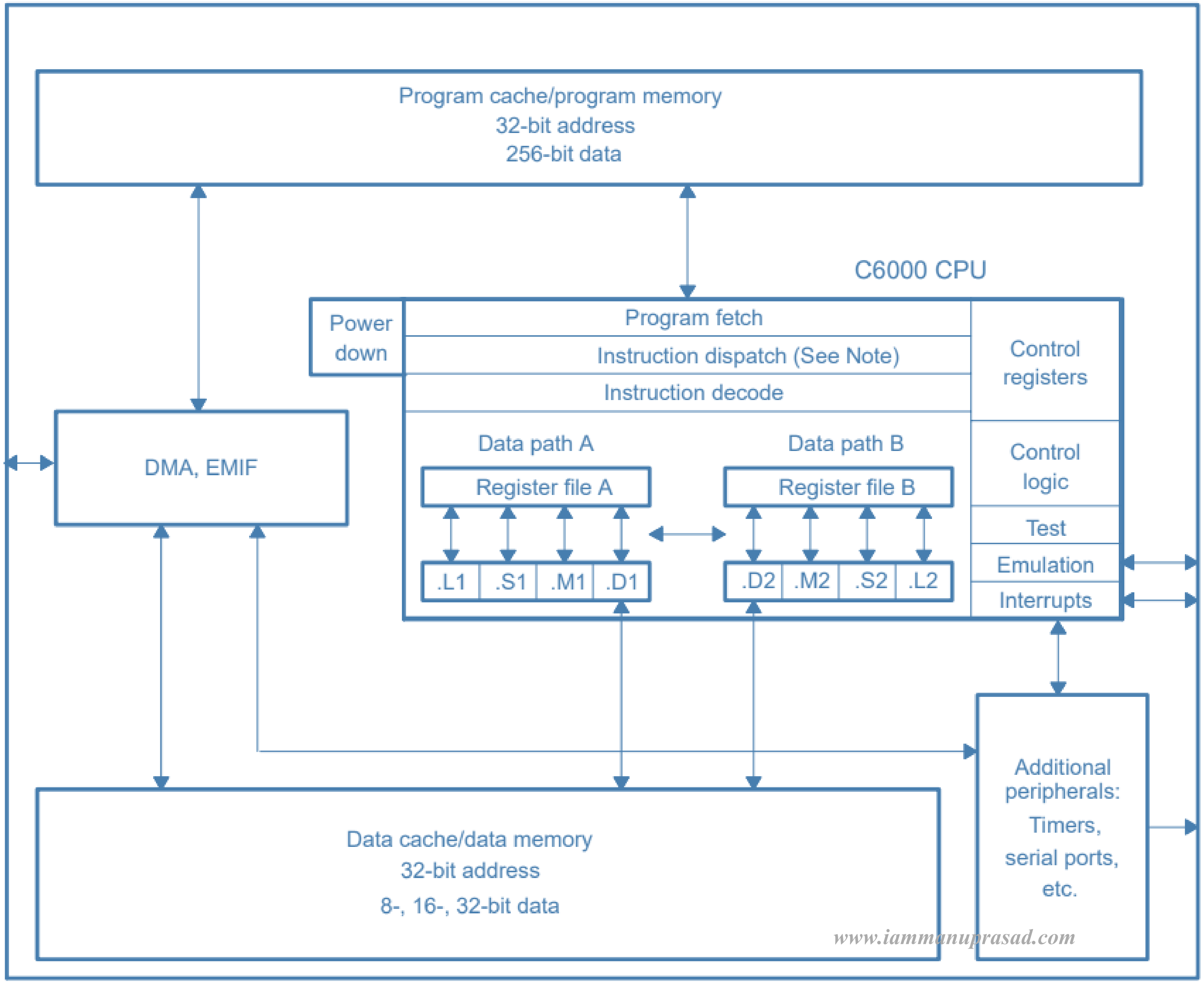
TMS320C67xx DSP Architecture

- **Central Processing Unit (CPU)**
  - Program fetch unit
  - Instruction dispatch unit
  - Instruction decode unit
  - Two data paths, each with four functional units
  - 32 32-bit registers
  - Control registers
  - Control logic
  - Test, emulation, and interrupt logic



TMS320C67xx DSP Architecture

- **DMA** – Direct memory Access
- **EMIF** – External Memory interface
- Program memory/Program cache
- Data memory /Data cache



TMS320C67xx DSP Architecture

- DMA Controller (C6701 DSP only) transfers data between address ranges in the memory map without intervention by the CPU. The DMA controller has four programmable channels and a fifth auxiliary channel.
- EDMA Controller performs the same functions as the DMA controller. The EDMA has 16 programmable channels, as well as a RAM space to hold multiple configurations for future transfers.
- HPI is a parallel port through which a host processor can directly access the CPU's memory space. The host device has ease of access because it is the master of the interface. The host and the CPU can exchange information via internal or external memory. In addition, the host has direct access to memory-mapped peripherals.
- Expansion bus is a replacement for the HPI, as well as an expansion of the EMIF. The expansion provides two distinct areas of functionality (host port and I/O port) which can co-exist in a system. The host port of the expansion bus can operate in either asynchronous slave mode, similar to the HPI, or in synchronous master/slave mode. This allows the device to interface to a variety of host bus protocols. Synchronous FIFOs and asynchronous peripheral I/O devices may interface to the expansion bus.

TMS320C67xx DSP Architecture

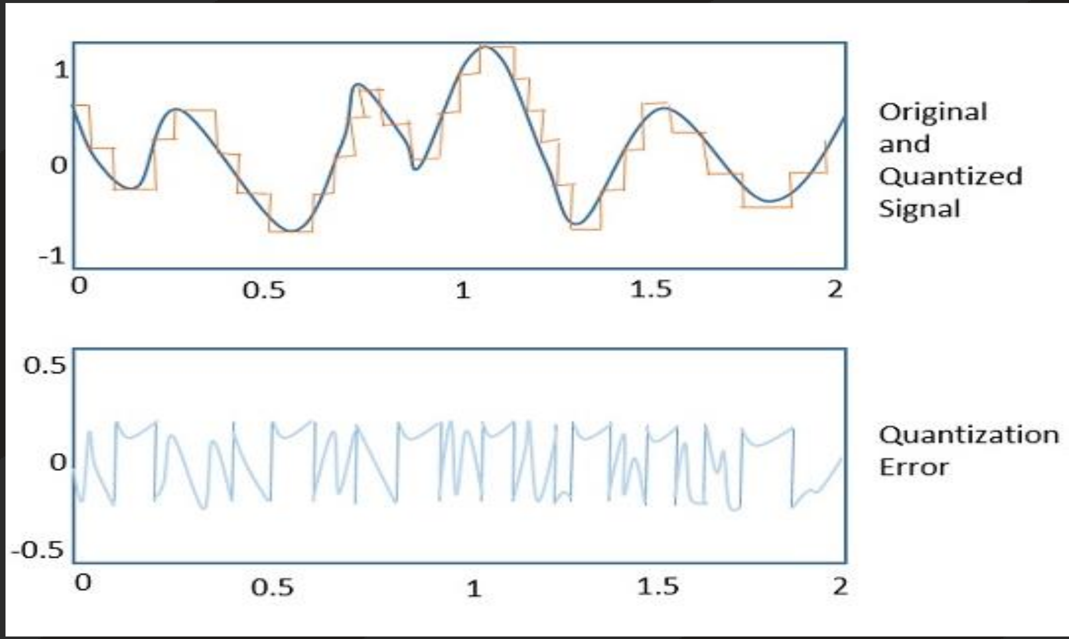
- McBSP (multichannel buffered serial port) is based on the standard serial port interface found on the TMS320C2000 and TMS320C5000 devices. In addition, the port can buffer serial samples in memory automatically with the aid of the DMA/EDNA controller. It also has multichannel capability compatible with the T1, E1, SCSA, and MVIP networking standards.
- Timers in the C6000 devices are two 32-bit general-purpose timers used for these functions:
  - Time events
  - Count events
  - Generate pulses
  - Interrupt the CPU
  - Send synchronization events to the DMA/EDMA controller.
- Power-down logic allows reduced clocking to reduce power consumption. Most of the operating power of CMOS logic dissipates during circuit switching from one logic state to another. By preventing some or all of the chip's logic from switching, you can realize significant power savings without losing any data or operational context.

Finite Word length Effects

- In the design of FIR Filters, the filter coefficients are determined by the system transfer functions. These filters co-efficient are quantized/truncated while implementing DSP System because of finite length registers.
- Only Finite numbers of bits are used to perform arithmetic operations. Typical word length is 16 bits, 24 bits, 32 bits etc.
- This finite word length introduces an error which can affect the performance of the DSP system.
  - Input quantization error
  - Co-efficient quantization error
  - Overflow & round off error (Product Quantization error)

Quantization Error

- The effect of error introduced by a signal process depend upon number of factors including the.
  - Type of arithmetic
  - Quality of input signal
  - Type of algorithm implemented
- For any system, during its functioning, there is always a difference in the values of its input and output. The processing of the system results in an error, which is the difference of those values. The difference between an input value and its quantized value is called a Quantization Error.



Input quantization error

- The conversion of continuous-time input signal into digital value produces an error which is known as input quantization error. This error arises due to the representation of the input signal by a fixed number of digits in A/D conversion process

$$e(n) = x_q(n) - x(n)$$

$x_q(n) \rightarrow sample\ quatised\ value$

$x(n) \rightarrow input\ value$

For example, let $x(n) = (0.70)_{10} = (0.10110011...)_2$

add

After rounding $x(n)$ to 3 bits we have

$$\begin{aligned} x_q(n) &= 0.101 \\ &\quad\;\; 1 \\ &\overline{0.110} \\ &= (0.75)_{10} \end{aligned}$$

add

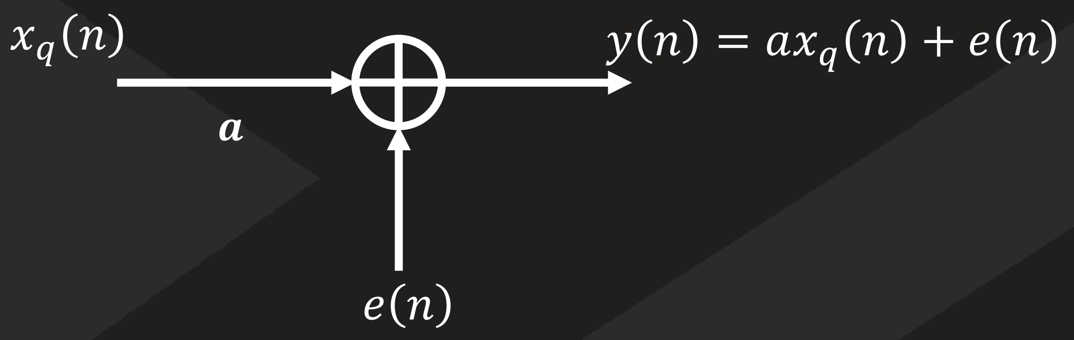
Now the error

$$e(n) = x_q(n) - x(n) = 0.05$$

which satisfies the inequality.

Product Quantization error

- In fixed point arithmetic the product of two *b-bit* numbers results in *2b* bits long. In DSP applications it is necessary to round this product to *b-bit* number which produce an error known as product quantization error or product round off noise



- The multiplication is modelled as an infinite precision multiplier followed by an adder where round off noise is added to the product so that overall result equals some quantization level

1. For any n, the error sequence $e(n)$ is uniformly distributed over the range $\frac{-q}{2}$ and $\frac{q}{2}$. This implies that mean value of $e(n)$ is zero and its variance is $\sigma_e^2 = \frac{2^{-2b}}{12}$
2. The error sequence $e(n)$ is a stationary white noise sequence.
3. The error sequence $e(n)$ is uncorrelated with the signal sequence $x(n)$. Thus each noise source is modeled as a discrete stationary white random process with a power density spectrum of $\frac{2^{-2b}}{12}$.

Coefficient quantization error

- In the design of a digital filter the coefficients are evaluated with infinite precision.
- But when they are quantized, the frequency response of the actual filter deviates from that which would have been obtained with an infinite word length representation and the filter may actually fail to meet the desired specifications.
- If the poles of the desired filter are close to the unit circle, then those of the filter with quantized coefficients my lie just outside the unit circle

Coefficient quantization error

- Consider a second order IIR filter with

$$H(z) = \frac{1.0}{(1-0.5z^{-1})(1-0.45z^{-1})}$$

find the effect on quantization on pole locations of the given system function in direct form and in cascade form . Take $b = 3$ bits.

Solution

Direct Form I

We can write $H(z) = \dfrac{1}{(1-0.95z^{-1}+0.225z^{-2})}$

$$(0.95)_{10} = (0.1111001...)_2$$
$$(-0.95)_{10} = (1.1111001...)_2$$

After truncation we have $(1.111)_2 = -0.875$. Similarly

$$(0.225)_{10} = (0.001110...)_2$$

After truncation we have $(0.001)_2 = 0.125$

$$\text{So} \quad H(z) = \frac{1}{(1-0.875z^{-1}+0.125z^{-2})}$$

Cascade form

$$H(z) = \frac{1}{(1-0.5z^{-1})(1-0.45z^{-1})}$$
$$(-0.5)_{10} = (1.100)_2$$
$$(-0.45)_{10} = (1.01110\ldots)_2$$

After truncation we have $(1.011)_2 = (-0.375)_{10}$

$$\text{So} \quad H(z) = \frac{1}{(1-0.5z^{-1})(1-0.375z^{-1})}$$